ALGEBRA IN SUPEREXTENSIONS OF TWINIC GROUPS

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ABSTRACT. Given a group X we study the algebraic structure of the compact right-topological semigroup $\lambda(X)$ consisting of maximal linked systems on X. This semigroup contains the semigroup $\beta(X)$ of ultrafilters as a closed subsemigroup. We construct a faithful representation of the semigroup $\lambda(X)$ in the semigroup $\mathsf{P}(X)^{\mathsf{P}(X)}$ of all self-maps of the power-set $\mathsf{P}(X)$ and show that the image of $\lambda(X)$ in $\mathsf{P}(X)^{\mathsf{P}(X)}$ coincides with the semigroup $\mathrm{End}_\lambda(\mathsf{P}(X))$ of all functions $f : \mathsf{P}(X) \to \mathsf{P}(X)$ that are equivariant, monotone and symmetric in the sense that $f(X \setminus A) = X \setminus f(A)$ for all $A \subset X$. Using this representation we describe the minimal ideal $K(\lambda(X))$ and minimal left ideals of the superextension $\lambda(X)$ of a twinic group X. A group X is called *twinic* if it admits a left-invariant ideal $\mathcal{I} \subset \mathsf{P}(X)$ such that $xA =_\mathcal{I} yA$ for any subset $A \subset X$ and points $x, y \in X$ with $xA \subset_\mathcal{I} X \setminus A \subset_\mathcal{I} yA$. The class of twinic groups includes all amenable groups and all groups with periodic commutators but does not include the free group F_2 with two generators.

We prove that for an Abelian group X (admitting no epimorphism onto the quasi-cyclic group C_{2^∞}) each minimal left ideal of the superextension $\lambda(X)$ is algebraically (and topologically) isomorphic to the product $\prod_{1 \le k \le \infty}(C_{2^k} \times 2^{2^{k-1}-k})^{\eta(X, C_{2^k})}$ where $\eta(X, C_{2^k})$ is the number of subgroups $H \subset X$ such that the quotient group X/H is isomorphic to the (quasi)cyclic group C_{2^k} of order 2^k; here the cardinal $2^{2^{k-1}-k}$ (equal to continuum if $k = \infty$) is endowed with the discrete topology and left-zero multiplication.

Applying this result to the group $\mathbb{Z}$ of integers, we prove that each minimal left ideal of $\lambda(\mathbb{Z})$ is topologically isomorphic to $2^\omega \times \prod_{k=1}^\infty C_{2^k}$ where the Cantor cube 2^ω is endowed with a left zero multiplication. Consequently, all subgroups in the minimal ideal $K(\lambda(\mathbb{Z}))$ of $\lambda(\mathbb{Z})$ are profinite abelian groups. On the other hand, the superextension $\lambda(\mathbb{Z})$ contains an isomorphic topological copy of each second countable profinite topological semigroup. This results contrasts with the famous Zelenuk Theorem saying that the semigroup $\beta(\mathbb{Z})$ contains no finite subgroups. At the end of the paper we describe the structure of minimal left ideals of finite groups X of order $|X| \le 15$.

Contents

1. INTRODUCTION

After discovering a topological proof of Hindman theorem [9] (see [11, p.102], [10]), topological methods become a standard tool in the modern combinatorics of numbers, see [11], [16]. The crucial point is that any semigroup operation defined on a discrete space X can be extended to a right-topological semigroup operation on $\beta(X)$, the Stone-Čech compactification of X. The extension of the operation from X to $\beta(X)$ can be defined by the simple formula:

$$
\text{(1)} \qquad \mathcal{A} \circ \mathcal{B} = \big\{ A \subset X : \{ x \in X : x^{-1}A \in \mathcal{B} \} \in \mathcal{A} \big\},
$$

The Stone-Čech compactification $\beta(X)$ of X is the subspace of the double power-set $\mathsf{P}^2(X) = \mathsf{P}(\mathsf{P}(X))$, which can be identified with the Cantor discontinuum $\{0,1\}^{\mathsf{P}(X)}$ and endowed with the compact Hausdorff topology of the Tychonov product. It turns out that the formula (1) applied to arbitrary families $\mathcal{A}, \mathcal{B} \in \mathsf{P}^2(X)$ of subsets of a group X still defines a binary operation $\circ : \mathsf{P}^2(X) \times \mathsf{P}^2(X) \to \mathsf{P}^2(X)$ that turns the double power-set $\mathsf{P}^2(X)$ into a compact Hausdorff right-topological semigroup that contains $\beta(X)$ as a closed subsemigroup.

The semigroup $\beta(X)$ lies in a bit larger subsemigroup $\lambda(X) \subset \mathsf{P}^2(X)$ consisting of all maximal linked systems on X. We recall that a family $\mathcal{L}$ of subsets of X is

- *linked* if any sets $A, B \in \mathcal{L}$ have non-empty intersection $A \cap B \neq \emptyset$;
- *maximal linked* if $\mathcal{L}$ coincides with each linked system $\mathcal{L}'$ on X that contains $\mathcal{L}$.

The space $\lambda(X)$ is well-known in General and Categorial Topology as the *superextension* of X, see [13], [18].

The thorough study of algebraic properties of the superextensions of groups was started in [2] and continued in [3] and [4]. In particular, in [4] we proved that the minimal left ideals of the superextension $\lambda(\mathbb{Z})$ are metrizable topological semigroups. In this paper we shall extend this result to the superextensions $\lambda(X)$ of all finitely-generated abelian groups X (more generally, abelian groups admitting no homomorphism onto the quasi-cyclic 2-group).

The results obtained in this paper completely reveal the topological and algebraic structure of the minimal ideal and minimal left ideals of the superextension of a twinic group. A group X is defined to be *twinic* if it admits a left-invariant ideal $\mathcal{I}$ of subsets of X such that for any subset $A \subset X$ with $xA \subset_{\mathcal{I}} X \setminus A \subset_{\mathcal{I}} yA$ for some $x, y \in X$ we have $A =_{\mathcal{I}} B$. Here the symbol $A \subset_{\mathcal{I}} B$ means that $A \setminus B \in \mathcal{I}$ and $A =_{\mathcal{I}} B$ means that $A \subset_{\mathcal{I}} B$ and $B \subset_{\mathcal{I}} A$. The class of twinic groups contains all amenable groups and all groups with periodic commutators (in particular, all torsion groups), but does not contain the free group with two generators F_2.

For Abelian groups the principal results are relatively simple and can be formulated here, in the introduction. Yet, some notation should be fixed.

By $C_{2^k} = \{ z \in \mathbb{C} : z^{2^k} = 1 \}$ we denote the cyclic group of order 2^k. The union $C_{2^\infty} = \bigcup_{k=1}^\infty C_{2^k}$ is called the *quasicyclic 2-group*.

For a group X by $\eta(X, C_{2^k})$ we denote the number of normal subgroups $H \subset X$ with quotient X/H isomorphic to C_{2^k}. It is easy to see that for $k \in \mathbb{N}$

$$
\eta(X, C_{2^k}) = \frac{\hom(X, C_{2^k}) - \hom(X, C_{2^{k-1}})}{2^{k-1}}
$$

where $\hom(X, C_{2^k})$ is the group of homomorphisms from X into C_{2^k}.

The following theorem is a particular case of Theorems 17.1 and 17.4.

Theorem 1.1. *If X is an abelian group (admitting no homomorphism onto C_{2^∞}), then*

(1) *each minimal left ideal of $\lambda(X)$ is algebraically (and topologically) isomorphic to*

$$
\prod_{1 \leq k \leq \infty} (C_{2^k} \times E_k)^{\eta(X, C_{2^k})}
$$

where each E_k is a left zero semigroup of cardinality $2^{2^{k-1}-k}$ if $k < \infty$ and of cardinality continuum if $k = \infty$;

(2) *the semigroup $\lambda(X)$ contains a principal left ideal, which is algebraically (and topologically) isomorphic to*

$$
\prod_{1 \leq k \leq \infty} (C_{2^k} \wr E_k^{E_k})^{\eta(X, C_{2^k})}
$$

where each $E_k^{E_k}$ is the topological semigroup of all self-maps of the discrete space E_k;

(3) each maximal subgroup of the minimal ideal of $\lambda(X)$ is algebraically (and topologically) isomorphic to
$$\prod_{1 \leq k \leq \infty} (C_{2^k})^{\eta(X, C_{2^k})}.$$

If the group X is finitely generated, then

(4) all minimal left ideals of the superextension $\lambda(X)$ are compact metrizable topological semigroups.

2. Right-topological semigroups

In this section we recall some information from [11] related to right-topological semigroups. By definition, a right-topological semigroup is a topological space S endowed with a semigroup operation $* : S \times S \to S$ such that for every $a \in S$ the right shift $r_a : S \to S$, $r_a : x \mapsto x * a$, is continuous. If the semigroup operation $* : S \times S \to S$ is (separately) continuous, then $(S, *)$ is a (semi-)topological semigroup.

From now on, S is a compact Hausdorff right-topological semigroup. We shall recall some known information concerning ideals in S, see [11].

A non-empty subset I of S is called a left (resp. right) ideal if $SI \subset I$ (resp. $IS \subset I$). If I is both a left and right ideal in S, then I is called an ideal in S. Observe that for every $x \in S$ the set $SxS = \{sxt : s, t \in S\}$ (resp. $Sx = \{sx : s \in S\}$, $xS = \{xs : s \in S\}$) is an ideal (resp. left ideal, right ideal) ideal in S. Such an ideal is called principal. An ideal $I \subset S$ is called minimal if any ideal of S that lies in I coincides with I. By analogy we define minimal left and right ideals of S. It is easy to see that each minimal left (resp. right) ideal I is principal. Moreover, $I = Sx$ (resp. $I = xS$) for each $x \in I$. This simple observation implies that each minimal left ideal in S, being principal, is closed in S. By [11, 2.6], each left ideal in S contains a minimal left ideal. The union $K(S)$ of all minimal left ideals of S coincides with the minimal ideal of S, [11, 2.8].

All minimal left ideals of S are mutually homeomorphic and all maximal groups of the minimal ideal $K(S)$ are algebraically isomorphic. Moreover, if two maximal groups lie in the same minimal right ideal, then they are topologically isomorphic.

An element z of a semigroup S is called a right zero (resp. a left zero) in S if $xz = z$ (resp. $zx = z$) for all $x \in S$. It is clear that $z \in S$ is a right (left) zero in S if and only if the singleton $\{z\}$ is a left (right) ideal in S.

We shall often use the following known fact, see [4, Lemma 1.1].

Proposition 2.1. If a homomorphism $h : S \to S'$ between two semigroups is injective on some minimal left ideal of S, then h is injective on each minimal left ideal of S.

An element $e \in S$ is called an idempotent if $ee = e$. By the Ellis Theorem [11, 2.5], the set $E(S)$ of idempotents of any compact right-topological semigroup is not empty. For every idempotent e the set
$$H(e) = \{x \in S : \exists x^{-1} \in S \ (xx^{-1}x = x, \ x^{-1}xx^{-1} = x^{-1}, \ xx^{-1} = e = x^{-1}x)\}$$
is a maximal subgroup of S containing e.

By [11, 1.48], for an idempotent $e \in E(S)$ the following conditions are equivalent:

- $e \in K(S)$;
- $K(S) = SeS$;
- Se is a minimal left ideal in S;
- eS is a minimal right ideal in S;
- eSe is a subgroup of S.

An idempotent e satisfying the above equivalent conditions will be called a minimal idempotent in S. By [11, 1.64], for any minimal idempotent $e \in S$ the set $E(Se) = E(S) \cap Se$ of idempotents of the minimal left ideal Se is a semigroup of left zeros, which means that $xy = x$ for all $x, y \in E(Se)$. By the Rees-Suschkewitsch Structure Theorem (see [11, 1.64]) the map
$$\varphi : E(Se) \times H(e) \to Se, \ \varphi : (x, y) \mapsto xy,$$
is an algebraic isomorphism of the corresponding semigroups. If the minimal left ideal Se is a topological semigroup, then φ is a topological isomorphism.

Now we see that all the information on the algebraic (and sometimes topological) structure of the minimal left ideal Se is encoded in the properties of the left zero semigroup $E(Se)$ and the maximal group $H(e)$.

3. ACTS AND THEIR ENDOMORPHISM MONOIDS

In this section we survey the information on acts that will be widely used in this paper for describing the algebraic structure of minimal left ideals of the superextensions of groups.

Following the terminology of [12] by an *act* we understand a set X endowed with a left action $\cdot : H \times X \to X$ of a group H called the *structure group* of the act. The action should satisfy two axioms: $1x = x$ and $g(hx) = (gh)x$ for all $x \in X$ and $g, h \in H$. Acts with the structure group H will be called *H-acts* or *H-spaces*.

An act X is called *free* if the stabilizer $\mathrm{Fix}(x) = \{h \in H : hx = x\}$ of each point $x \in X$ is trivial. For a point $x \in X$ by $[x] = \{hx : h \in H\}$ we denote its *orbit* and by $[X] = \{[x] : x \in X\}$ the orbit space of the act X. More generally, for each subset $A \subset X$ we put $[A] = \{[a] : a \in X\}$.

A function $f : X \to Y$ between two H-acts is called *equivariant* if $f(hx) = hf(x)$ for all $x \in X$ and $h \in H$. A function $f : X \to Y$ is called an *isomorphism* of the H-acts X and Y if it is bijective and equivariant. An equivariant self-map $f : X \to X$ is called an *endomorphism* of the H-act X. If f is bijective, then f is an *automorphism* of X.

The set $\mathrm{End}(X)$ of endomorphisms of an H-act X, endowed with the operation of composition of functions, is a monoid called the *endomorphism monoid* of X.

Each free H-act X is isomorphic to the product $H \times [X]$ endowed with the action $h \cdot (x, y) = (hx, y)$. For such an act the semigroup $\mathrm{End}(X)$ is isomorphic to the wreath product $H \wr [X]^{[X]}$ of the group H and the semigroup $[X]^{[X]}$ of all self-maps of the orbit space $[X]$.

The wreath product $H \wr A^A$ of a group H and the semigroup A^A of self-maps of a set A is defined as the semidirect product $H^A \rtimes A^A$ of the A-th power of H with A^A, endowed with the semigroup operation $(h, f) * (h', f') = (h'', f'')$ where $f'' = f \circ f'$ and $h''(\alpha) = h(f'(\alpha)) \cdot h'(\alpha)$ for $\alpha \in A$. For any subsemigroup $S \subset A^A$ the subset $H \wr S = \{(h, f) \in H^A \rtimes A^A : f \in S\}$ is called the *wreath product* of H and S. If both H and S are groups, then their wreath product $H \wr S$ is a group.

Denote by $A^{1_A} \subset A^A$ the subsemigroup consisting of constant functions and observe that A^{1_A} is isomorphic to the semigroup A endowed with the left zero multiplication $xy = x$. Observe that the maximal subgroup of A^A containing the identity self-map of A coincides with the group S_A of all bijective functions $f : A \to A$.

Theorem 3.1. *Let H be a group and X be a free H-act. Then*

(1) *the semigroup $\mathrm{End}(X)$ is isomorphic to the wreath product $H \wr [X]^{[X]}$;*
(2) *the minimal ideal $K(\mathrm{End}(X))$ of $\mathrm{End}(X)$ coincides with the set $\{f \in \mathrm{End}(X) : \forall x \in f(X)\ f(X) \subset [x]\}$;*
(3) *Each minimal left ideal of $\mathrm{End}(X)$ is isomorphic to $H \times [X]$ where $[X]$ is endowed with the left zero multiplication;*
(4) *for each idempotent $f \in \mathrm{End}(X)$ the maximal subgroup $H(f) \subset \mathrm{End}(X)$ is isomorphic to $H \wr S_{[f(X)]}$;*
(5) *for each idempotent $f \in K(\mathrm{End}(X))$ the maximal group $H(f) = f \cdot \mathrm{End}(X) \cdot f$ is isomorphic to H.*

Proof. 1. Let $\pi : X \to [X]$, $\pi : x \mapsto [x]$, denote the orbit map and $s : [X] \to X$ be a section of π, which means that $\pi \circ s([x]) = [x]$ for all $[x] \in [X]$.

Observe that each equivariant map $f : X \to X$ induces a well-defined map $[f] : [X] \to [X]$, $[f] : [x] \mapsto [f(x)]$, of the orbit spaces. Since the action of H on X is free, for every orbit $[x] \in [X]$ we can find a unique point $f_H([x]) \in H$ such that $f \circ s([x]) = (f_H([x]))^{-1} \cdot s([f(x)])$.

We claim that the map

$$\Psi : \mathrm{End}(X) \to H \wr [X]^{[X]}, \quad \Psi : f \mapsto (f_H, [f]),$$

is a semigroup isomorphism.

First we check that the map Ψ is a homomorphism. Pick any two equivariant functions $f, g \in \mathrm{End}(X)$ and consider their images $\Psi(f) = (f_H, [f])$ and $\Psi(g) = (g_H, [g])$ in $H \wr [X]^{[X]}$. Consider also the composition $f \circ g$ and its image $\Psi(f \circ g) = ((f \circ g)_H, [f \circ g])$. We claim that

$$((f \circ g)_H, [f \circ g]) = (f_H, [f]) * (g_H, [g]) = ((f_H \circ [g]) \cdot g_H, [f] \circ [g]).$$

The equality $[f \circ g] = [f] \circ [g]$ is clear. To prove that $(f \circ g)_H = (f_H \circ [g]) \cdot g_H$, take any orbit $[x] \in [X]$. It follows from the definition of $(f \circ g)_H([x])$ that

$$((f \circ g)_H([x]))^{-1} \cdot s([f \circ g(x)]) = (f \circ g) \circ s([x]) = f(g \circ s([x])) =$$
$$f\big((g_H([x]))^{-1} \cdot s([g(x)])\big) = (g_H([x]))^{-1} \cdot f \circ s([g(x)]) =$$
$$(g_H([x]))^{-1} \cdot (f_H([g(x)]))^{-1} \cdot s([f \circ g(x)]) =$$
$$(f_H \circ [g]([x]) \cdot g_H([x]))^{-1} \cdot s([f \circ g(x)])$$

which implies the desired equality $(f \circ g)_H = (f_H \circ [g]) \cdot g_H$.

Next, we show that the homomorphism Ψ is injective. Given two equivariant functions $f, g \in \mathrm{End}(X)$ with $(f_H, [f]) = \Psi(f) = \Psi(g) = (g_H, [g])$, we need to show that $f = g$. Observe that for every orbit $[x] \in [X]$ we get

$$f(s([x])) = (f_H([x]))^{-1} \cdot s \circ [f]([x])) = (g_H([x]))^{-1} \cdot s \circ [g]([x]) = g(s([x])).$$

Now for each $x \in X$ we can find a unique $h \in H$ with $x = h \cdot s([x])$ and apply the equivariantness of the functions f, g to conclude that

$$f(x) = f(h \cdot s([x]) = h \cdot f(s([x])) = h \cdot g(s([x])) = g(h \cdot s([x])) = g(x).$$

Finally, we show that Ψ is surjective. Given any pair $(h, g) \in H \wr [X]^{[X]} = H^{[X]} \times [X]^{[X]}$, we define an equivariant function $f \in \mathrm{End}(X)$ with $(h, g) = (f_H, [f])$ as follows. Given any $x \in X$ find a unique $y \in H$ with $x = y \cdot s([x])$ and let

$$f(x) = y \cdot h([x])^{-1} \cdot s(g([x])).$$

This formula determines a well-defined equivariant function $f : X \to X$ with $\Psi(f) = (h, g)$. Therefore, $\Psi : \mathrm{End}(X) \to H \wr [X]^{[X]}$ is a semigroup isomorphism.

2. Observe that the set $\mathcal{I} = \{f \in \mathrm{End}(X) : \{[f(x)] : x \in X\}$ is a singleton$\}$ is an ideal in $\mathrm{End}(X)$. We claim that each subideal $\mathcal{J} \subset \mathcal{I}$ coincides with $\mathcal{I}$, which will imply that $\mathcal{I}$ is a minimal ideal of $\mathrm{End}(X)$. Take any functions $f \in \mathcal{I}$ and $g \in \mathcal{J}$. Find an orbit $[x] \in [X]$ such that $[f]([X]) = \{[x]\}$. Since the restriction $g|[x] : [x] \to [g(x)]$ is bijective and equivariant, so is its inverse $(g|[x])^{-1} : [g(x)] \to [x]$. Extend this equivariant map to any equivariant map $h : X \to X$. Then

$$f = h \circ g \circ f \in \mathcal{J} \circ g \circ \mathcal{J} \subset \mathcal{J}.$$

3. Take any idempotent $f \in K(\mathrm{End}(X))$ and consider the minimal left ideal $\mathrm{End}(X) \cdot f$. Find a point $z \in X$ such that $f([X]) = \{[z]\}$ and let $Z = f^{-1}(z)$. It follows that the set Z meets each orbit $[x]$, $x \in X$, at a single point. So, we can define a unique section $s : [X] \to Z \subset X$ of the orbit map $X \to [X]$ such that $f \circ s([X]) = \{z\}$.

To each equivariant map $g \in \mathrm{End}(X)$ assign a unique element $g_H \in H$ such that $g(x) = g_H^{-1} \cdot s([g(x)])$. It is easy to check that the map

$$\Phi : \mathrm{End}(X) \cdot f \to H \times [X], \ \Phi : g \mapsto (g_H, [g]([x])),$$

is a semigroup homomorphism where the orbit space $[X]$ is endowed with the left zero multiplication.

4. Take any idempotent $f \in \mathrm{End}(X)$ and consider the surjective semigroup homomorphism $\mathrm{pr} : \mathrm{End}(X) \to [X]^{[X]}$, $\mathrm{pr} : g \mapsto [g]$. It follows that $[f]$ is an idempotent of the semigroup $[X]^{[X]}$ and the image $\mathrm{pr}(H(f))$ of the maximal group $H(f)$ is a subgroup of $[X]^{[X]}$. It is easy to see that the maximal subgroup $H[f]$ of the idempotent $[f]$ in $[X]^{[X]}$ coincides with $S_{[f(X)]} \cdot [f]$. The preimage $\mathrm{pr}^{-1}(H[f])$ of the maximal subgroup $H[f] = S_{[f(X)]} \cdot f$ is isomorphic to the wreath product $H \wr H[f]$ and hence is a group. Now the maximality of $H(f)$ guarantees that $H(f) = \mathrm{pr}^{-1}(H[f])$ and hence $H(f)$ is isomorphic to $H \wr S_{[f(X)]}$.

6. If $f \in K(\mathrm{End}(X))$ is an idempotent, then $[f(X)]$ is a singleton by the second item. By the preceding item the maximal group $H(f)$ is isomorphic to $H \wr S_{[f(X)]}$, which is isomorphic to the group H since $[f(X)]$ is a singleton. $\qquad\square$

For each group X the power-set $\mathsf{P}(X)$ will be considered as an act endowed with the left action

$$\cdot : X \times \mathsf{P}(X) \to \mathsf{P}(X), \quad \cdot : (x, A) \mapsto xA = \{xa : a \in A\},$$

of the group X. This X-act $\mathsf{P}(X)$ and its endomorphism monoid $\mathrm{End}(\mathsf{P}(X))$ play a crucial role in our considerations.

4. THE FUNCTION REPRESENTATION OF THE SEMIGROUP $\mathsf{P}^2(X)$

In this section given a group X we construct a topological isomorphism

$$\Phi : \mathsf{P}^2(X) \to \mathrm{End}(\mathsf{P}(X))$$

called the *function representation* of the semigroup $\mathsf{P}^2(X)$ in the endomorphism monoid of the X-act $\mathsf{P}(X)$. We recall that the double power-set $\mathsf{P}^2(X)$ of the group X is endowed with the binary operation

$$\mathcal{A} \circ \mathcal{B} = \left\{ A \subset X : \{x \in X : x^{-1}A \in \mathcal{B}\} \in \mathcal{A} \right\}.$$

The isomorphism Φ assigns to each family $\mathcal{A}$ of subsets of X the function

$$\Phi_{\mathcal{A}} : \mathsf{P}(X) \to \mathsf{P}(X), \quad \Phi_{\mathcal{A}} : A \mapsto \{x \in X : x^{-1}A \in \mathcal{A}\},$$

called the *function representation* of $\mathcal{A}$.

In the following theorem by e we denote the neutral element of the group X.

Theorem 4.1. *For any group X the map $\Phi : \mathsf{P}^2(X) \to \mathrm{End}(\mathsf{P}(X))$ is a topological isomorphism with inverse $\Phi^{-1} : \varphi \mapsto \{A \subset X : e \in \varphi(A)\}$.*

Proof. First observe that for any family $\mathcal{A} \in \mathsf{P}^2(X)$ the function $\Phi_{\mathcal{A}}$ is equivariant, because

$$\Phi_{\mathcal{A}}(xA) = \{y \in X : y^{-1}xA \in \mathcal{A}\} = \{xz : z^{-1}A \in \mathcal{A}\} = x\,\Phi_{\mathcal{A}}(A)$$

for any $x \in X$ and $A \subset X$. Thus the map $\Phi : \mathsf{P}^2(X) \to \mathrm{End}(\mathsf{P}(X))$ is well-defined.

To prove that Φ is a semigroup homomorphism, take two inclusion hyperspaces $\mathcal{X}, \mathcal{Y} \in \mathsf{P}^2(X)$ and let $\mathcal{Z} = \mathcal{X} \circ \mathcal{Y}$. We need to check that $\Phi_{\mathcal{Z}}(A) = \Phi_{\mathcal{X}} \circ \Phi_{\mathcal{Y}}(A)$ for every $A \subset X$. Observe that

$$\Phi_{\mathcal{Z}}(A) = \{z \in X : z^{-1}A \in \mathcal{Z}\} = \{z \in X : \{x \in X : x^{-1}z^{-1}A \in \mathcal{Y}\} \in \mathcal{X}\} =$$
$$= \{z \in X : \Phi_{\mathcal{Y}}(z^{-1}A) \in \mathcal{X}\} = \{z \in X : z^{-1}\Phi_{\mathcal{Y}}(A) \in \mathcal{X}\} = \Phi_{\mathcal{X}}(\Phi_{\mathcal{Y}}(A)) = \Phi_{\mathcal{X}} \circ \Phi_{\mathcal{Y}}(A).$$

To see that the map Φ is injective, take any two distinct families $\mathcal{A}, \mathcal{B} \in \mathsf{P}^2(X)$. Without loss of generality, $\mathcal{A} \setminus \mathcal{B}$ contains some set $A \subset X$. It follows that $e \in \Phi_{\mathcal{A}}(A)$ but $e \notin \Phi_{\mathcal{B}}(A)$ and hence $\Phi_{\mathcal{A}} \neq \Phi_{\mathcal{B}}$.

To see that the map Φ is surjective, take any equivariant function $\varphi : \mathsf{P}(X) \to \mathsf{P}(X)$ and consider the family $\mathcal{A} = \{A \subset X : e \in \varphi(A)\}$. It follows that for every $A \in \mathsf{P}(X)$

$$\Phi_{\mathcal{A}}(A) = \{x \in X : x^{-1}A \in \mathcal{A}\} = \{x \in X : e \in \varphi(x^{-1}A)\} = \{x \in X : e \in x^{-1}\varphi(A)\} = \{x \in X : x \in \varphi(A)\} = \varphi(A).$$

To prove that $\Phi : \mathsf{P}^2(X) \to \mathrm{End}(\mathsf{P}(X)) \subset \mathsf{P}(X)^{\mathsf{P}(X)}$ is continuous we first define a convenient sub-base of the topology on the spaces $\mathsf{P}(X)$ and $\mathsf{P}(X)^{\mathsf{P}(X)}$. The product topology of $\mathsf{P}(X)$ is generated by the sub-base consisting of the sets

$$x^+ = \{A \subset X : x \in A\} \text{ and } x^- = \{A \subset X : x \notin A\}$$

where $x \in X$. On the other hand, the product topology on $\mathsf{P}(X)^{\mathsf{P}(X)}$ is generated by the sub-base consisting of the sets

$$\langle x, A \rangle^+ = \{f \in \mathsf{P}(X)^{\mathsf{P}(X)} : x \in f(A)\} \text{ and } \langle x, A \rangle^- = \{f \in \mathsf{P}(X)^{\mathsf{P}(X)} : x \notin f(A)\}$$

where $A \in \mathsf{P}(X)$ and $x \in X$.

Now observe that the preimage

$$\Phi^{-1}(\langle x, A \rangle^+) = \{\mathcal{A} \in \mathsf{P}^2(X) : x \in \Phi_{\mathcal{A}}(A)\} = \{\mathcal{A} \in \mathsf{P}^2(X) : x^{-1}A \in \mathcal{A}\}$$

is open in $\mathsf{P}^2(X)$. The same is true for the preimage

$$\Phi^{-1}(\langle x, A \rangle^-) = \{\mathcal{A} \in \mathsf{P}^2(X) : x \notin \Phi_{\mathcal{A}}(A)\} = \{\mathcal{A} \in \mathsf{P}^2(X) : x^{-1}A \notin \mathcal{A}\}$$

which also is open in $\mathsf{P}^2(X)$.

Since the spaces $\mathsf{P}^2(X) \cong \{0,1\}^{\mathsf{P}(X)}$ and $\mathrm{End}(\mathsf{P}(X)) \subset \mathsf{P}(X)^{\mathsf{P}(X)}$ are compact and Hausdorff, the continuity of the map Φ implies the continuity of its inverse Φ^{-1}. Consequently, $\Phi : \mathsf{P}^2(X) \to \mathrm{End}(\mathsf{P}(X))$ is a topological isomorphism of compact right-topological semigroup. $\qquad\square$

Remark 4.2. The functions representations $\Phi_{\mathcal{A}}$ of some families $\mathcal{A} \subset \mathsf{P}(X)$ have transparent topological interpretations. For example, if $\mathcal{A}$ is the filter of neighborhoods of the identity element e of a left-topological group X, then for any subset $A \subset X$ the set $\Phi_{\mathcal{A}}(A)$ coincides with the interior of a set $A \subset X$ while $\Phi_{\mathcal{A}^\perp}(A)$ with the closure of A in X!

Theorem 4.1 has a strategical importance because it allows us to translate (usually difficult) problems concerning the structure of the semigroup $\mathsf{P}^2(X)$ to (usually more tractable) problems about the endomorphism monoid $\mathrm{End}(\mathsf{P}(X))$. In particular, Theorem 4.1 implies "for free" that the binary operation on $\mathsf{P}^2(X)$ is associative and right-topological and hence $\mathsf{P}^2(X)$ indeed is a compact right-topological semigroup.

Now let us investigate the interplay between the properties of families $\mathcal{A} \in \mathsf{P}^2(X)$ and their function representations $\Phi_{\mathcal{A}}$.

Let us define a family $\mathcal{A} \subset \mathsf{P}(X)$ to be
- *monotone* if for any subsets $A \subset B \subset X$ the inclusion $A \in \mathcal{A}$ implies $B \in \mathcal{A}$;
- *left-invariant* if for any $A \in \mathcal{A}$ and $x \in X$ we get $xA \in \mathcal{A}$.

Respectively, a function $\varphi : \mathsf{P}(X) \to \mathsf{P}(X)$ is called
- *monotone* if $\varphi(A) \subset \varphi(B)$ for any subsets $A \subset B \subset X$;
- *symmetric* if $\varphi(X \setminus A) = X \setminus \varphi(A)$ for every $A \subset X$.

Proposition 4.3. *For an equivariant function $\varphi \in \mathrm{End}(\mathsf{P}(X))$ the family $\Phi^{-1}(\varphi) = \{A \subset X : e \in \varphi(A)\}$ is*
(1) *monotone if and only if φ is monotone;*
(2) *left-invariant if and only if $\varphi(\mathsf{P}(X)) \subset \{\emptyset, X\}$;*
(3) *maximal linked if and only if φ is monotone and symmetric.*

Proof. Let $\mathcal{A} = \Phi^{-1}(\varphi)$.

1. If φ is monotone, then for any sets $A \subset B$ with $A \in \mathcal{A}$ we get $e \in \varphi(A) \subset \varphi(B)$ and hence $B \in \mathcal{A}$, which means that the family $\mathcal{A}$ is monotone.

Now assume conversely that the family $\mathcal{A}$ is monotone and take any sets $A \subset B \subset X$. Note that for any $x \in X$ with $xA \in \mathcal{A}$ we get $xB \in \mathcal{A}$. Then
$$\varphi(A) = \{x \in X : x^{-1}A \in \mathcal{A}\} \subset \{x \in X : x^{-1}B \in \mathcal{A}\} = \varphi(B),$$
witnessing that the function φ is monotone.

2. If the family $\mathcal{A}$ is invariant, then for each $A \in \mathcal{A}$ we get $\varphi(A) = \{x \in X : x^{-1}A \in \mathcal{A}\} = X$ and for each $A \notin \mathcal{A}$ we get $\varphi(A) = \{x \in X : x^{-1}A \in \mathcal{A}\} = \emptyset$.

Now assume conversely that $\varphi(\mathsf{P}(X)) \subset \{\emptyset, X\}$. Then for each $A \in \mathcal{A}$ we get $e \in \varphi(A) = X$ and then for each $x \in X$, the equivariance of φ guarantees that $\varphi(xA) = x\varphi(A) = xX = X \ni e$ and thus $xA \in \mathcal{A}$, witnessing that the family $\mathcal{A}$ is invariant.

3. Assume that the family $\mathcal{A}$ is maximal linked. By the maximality, $\mathcal{A}$ is monotone. Consequently, its function representation φ is monotone. The maximal linked property of $\mathcal{A}$ guarantees that for any subset $A \subset X$ we get $(A \in \mathcal{A}) \Leftrightarrow (X \setminus A \notin \mathcal{A})$. Then
$$\varphi(X \setminus A) = \{x \in X : x^{-1}(X \setminus A) \in \mathcal{A}\} = \{x \in X : X \setminus x^{-1}A \in \mathcal{A}\} =$$
$$= \{x \in X : x^{-1}A \notin \mathcal{A}\} = X \setminus \{x \in X : x^{-1}A \in \mathcal{A}\} = X \setminus \varphi(A),$$
which means that the function φ is symmetric.

Now assuming that the function φ is monotone and symmetric, we shall show that the family $\mathcal{A} = \Phi^{-1}(\varphi)$ is maximal linked. The statement (1) guarantees that $\mathcal{A}$ is monotone. Assuming that $\mathcal{A}$ is not linked, we could find two disjoint sets $A, B \in \mathcal{A}$. Since $\mathcal{A}$ is monotone, we can assume that $B = X \setminus A$. Then $e \in \varphi(A) \cap \varphi(X \setminus A)$, which is impossible as $\varphi(X \setminus A) = X \setminus \varphi(A)$. Thus $\mathcal{A}$ is linked. To show that $\mathcal{A}$ is maximal linked, it suffices to check that for each subset $A \subset X$ either A or $X \setminus A$ belongs to $\mathcal{A}$. Since $\varphi(X \setminus A) = X \setminus \varphi(A)$, either $\varphi(A)$ or $\varphi(X \setminus A)$ contains the neutral element e of the group X. In the first case $A \in \mathcal{A}$ and in the second case $X \setminus A \in \mathcal{A}$. $\qquad\square$

Let us recall that the aim of this paper is the description of the structure of minimal left ideals of the superextension $\lambda(X)$ of a group X. Instead of the semigroup $\lambda(X)$ it will be more convenient to consider its isomorphic copy

$$\mathrm{End}_\lambda(\mathsf{P}(X)) = \Phi(\lambda(X)) \subset \mathrm{End}(\mathsf{P}(X))$$

called the *function representation of* $\lambda(X)$.

Proposition 4.3 implies

Corollary 4.4. *The function representation* $\mathrm{End}_\lambda(\mathsf{P}(X))$ *of* $\lambda(X)$ *consists of equivariant monotone symmetric functions* $\varphi : \mathsf{P}(X) \to \mathsf{P}(X)$.

In order to describe the structure of minimal left ideals of the semigroup $\mathrm{End}_\lambda(\mathsf{P}(X))$ we shall look for a relatively small subfamily $\mathsf{F} \subset \mathsf{P}(X)$ such that the restriction operator

$$R_\mathsf{F} : \mathrm{End}_\lambda(\mathsf{P}(X)) \to \mathsf{P}(X)^\mathsf{F}, \quad R_\mathsf{F} : \varphi \mapsto \varphi|\mathsf{F},$$

is injective on each minimal left ideal of the semigroup $\mathrm{End}_\lambda(\mathsf{P}(X))$.

Then the composition

$$\Phi_\mathsf{F} = R_\mathsf{F} \circ \Phi : \lambda(X) \to \mathsf{P}(X)^\mathsf{F}$$

will be injective on each minimal left ideal of the semigroup $\lambda(X)$. By Proposition 2.1, a homomorphism between semigroups is injective on each minimal left ideal if it is injective on some minimal left ideal. Such special minimal left ideal of the semigroup $\lambda(X)$ will be found in a left ideal of the form $\lambda^\mathcal{I}(X)$ for a suitable left-invariant ideal $\mathcal{I}$ of subsets of the group X.

A family $\mathcal{I}$ of subsets of X is called an *ideal* on X if

- $X \notin \mathcal{I}$;
- $A \cup B \in \mathcal{I}$ for any $A, B \in \mathcal{I}$;
- for any $A \in \mathcal{I}$ and $B \subset A$ we get $B \in \mathcal{I}$.

Such an ideal $\mathcal{I}$ is called *left-invariant* if $xA \in \mathcal{I}$ for all $A \in \mathcal{I}$ and $x \in X$. The smallest ideal on X is the trivial ideal $\{\emptyset\}$ containing only the empty set. The smallest non-trivial left-invariant ideal on an infinite group X is the ideal $[X]^{<\omega}$ of finite subsets of X. From now on we shall assume that $\mathcal{I}$ is a left-invariant ideal on a group X.

For subsets $A, B \subset X$ we write

- $A \subset_\mathcal{I} B$ if $A \setminus B \in \mathcal{I}$, and
- $A =_\mathcal{I} B$ if $A \subset_\mathcal{I} B$ and $B \subset_\mathcal{I} A$.

The additivity property of the ideal $\mathcal{I}$ implies that $=_\mathcal{I}$ is an equivalence relation on $\mathsf{P}(X)$.

A family $\mathcal{A}$ of subsets of X is defined to be $\mathcal{I}$-*free* if for any $A \in \mathcal{A}$ and a subset $B =_\mathcal{I} A$ of X we get $B \in \mathcal{A}$. Let us observe that a monotone family $\mathcal{A} \subset \mathsf{P}(X)$ is $\mathcal{I}$-free if and only if for any $A \in \mathcal{A}$ and $B \in \mathcal{I}$ we get $A \setminus B \in \mathcal{A}$.

Respectively, a function $\varphi : \mathsf{P}(X) \to \mathsf{P}(X)$ is called $\mathcal{I}$-*free* if $\varphi(A) = \varphi(B)$ for any subsets $A =_\mathcal{I} B$ of X.

Proposition 4.5. *A family* $\mathcal{A} \subset \mathsf{P}(X)$ *is* $\mathcal{I}$-*free if and only if so is its function representation* $\Phi_\mathcal{A} : \mathsf{P}(X) \to \mathsf{P}(X)$.

Proof. Assume that $\mathcal{A}$ is $\mathcal{I}$-free and take two subsets $A =_\mathcal{I} B$ of X. We need to show that $\Phi_\mathcal{A}(A) = \Phi_\mathcal{A}(B)$. The left-invariance of the ideal $\mathcal{I}$ implies that for every $x \in X$ we get $xA =_\mathcal{I} xB$ and hence $(xA \in \mathcal{A}) \Leftrightarrow (xB \in \mathcal{A})$. Then

$$\Phi_\mathcal{A}(A) = \{x \in X : x^{-1}A \in \mathcal{A}\} = \{x \in X : x^{-1}B \in \mathcal{A}\} = \Phi_\mathcal{A}(B).$$

Now assume conversely that the function representation $\Phi_\mathcal{A}$ is $\mathcal{I}$-free and take any subsets $A =_\mathcal{I} B$ with $A \in \mathcal{A}$. Then $e \in \Phi_\mathcal{A}(A) = \Phi_\mathcal{A}(B)$, which implies that $B \in \mathcal{A}$. $\square$

For an left-invariant ideal $\mathcal{I}$ on a group X let $\lambda^\mathcal{I}(X) \subset \lambda(X)$ be the subspace of $\mathcal{I}$-free maximal linked systems on X and $\mathrm{End}_\lambda^\mathcal{I}(\mathsf{P}(X)) \subset \mathrm{End}_\lambda(\mathsf{P}(X))$ be the subspace consisting of $\mathcal{I}$-free monotone symmetric endomorphisms of the X-act $\mathsf{P}(X)$. It is clear that for any functions $f, g : \mathsf{P}(X) \to \mathsf{P}(X)$ the composition $f \circ g$ is $\mathcal{I}$-free provided so is the function g. This trivial remark implies:

Proposition 4.6. *For any ideal $\mathcal{I}$ the function representation $\Phi : \lambda^{\mathcal{I}}(X) \to \operatorname{End}_{\lambda}^{\mathcal{I}}(\mathsf{P}(X))$ is a topological isomorphism between the closed left ideals $\lambda^{\mathcal{I}}(X)$ and $\operatorname{End}_{\lambda}^{\mathcal{I}}(\mathsf{P}(X))$ of the semigroups $\lambda(X)$ and $\operatorname{End}_{\lambda}(\mathsf{P}(X))$, respectively.*

The following lemma (combined with the Zorn Lemma) implies that the sets $\lambda^{\mathcal{I}}(X)$ and $\operatorname{End}_{\lambda}^{\mathcal{I}}(\mathsf{P}(X))$ are not empty.

Lemma 4.7. *Each maximal $\mathcal{I}$-free linked system $\mathcal{L}$ on X is maximal linked.*

Proof. We need to show that each set $A \subset X$ that meets all sets $L \in \mathcal{L}$ belongs to $\mathcal{L}$. We claim that $A \notin \mathcal{I}$. Otherwise, taking any subset $L \in \mathcal{L}$, we get $L \setminus A =_{\mathcal{I}} L$ and hence $L \setminus A$ belongs to $\mathcal{L}$, which is not possible as $L \setminus A$ misses the set A. Since $A \notin \mathcal{I}$ the $\mathcal{I}$-free family $\mathcal{A} = \{A' \subset X : A' =_{\mathcal{I}} A\}$ is linked.

We claim that the $\mathcal{I}$-free family $\mathcal{A} \cup \mathcal{L}$ is linked. Assuming the converse, we would find two disjoint sets $A' \in \mathcal{A}$ and $L \in \mathcal{L}$. Then $L \cap A =_{\mathcal{I}} L \cap A' =_{\mathcal{I}} \emptyset$ and hence the set $L \setminus A =_{\mathcal{I}} L$ belongs to $\mathcal{L}$, which is not possible as this set misses A.

Now we see that the family $\mathcal{A} \cup \mathcal{L}$, being $\mathcal{I}$-free and linked, coincides with the maximal $\mathcal{I}$-free linked system $\mathcal{L}$. Then $A \in \mathcal{A} \cup \mathcal{L} = \mathcal{L}$. $\qquad\square$

5. Twin and $\mathcal{I}$-twin subsets of groups X

In this section we start studying very interesting objects called twin sets. For an abelian (more generally, twinic) group X twin subsets of X form a subfamily $\mathsf{T} \subset \mathsf{P}(X)$ for which the function representation $\Phi_{\mathsf{T}} : \lambda(X) \to \operatorname{End}_{\lambda}(\mathsf{T})$ is injective on some left ideal of the superextension $\lambda(X)$.

For a subset A of a group X consider the following three subsets of X:

$$\operatorname{Fix}(A) = \{x \in X : xA = A\}, \quad \operatorname{Fix}^{-}(A) = \{x \in X : xA = X \setminus A\}, \quad \text{and} \quad \operatorname{Fix}^{\pm}(A) = \operatorname{Fix}(A) \cup \operatorname{Fix}^{-}(A).$$

Definition 5.1. A subset $A \subset X$ is defined to be
- *twin* if $xA = X \setminus A$ for some $x \in X$,
- *pretwin* if $xA \subset X \setminus A \subset yA$ for some points $x, y \in X$.

The families of twin and pretwin subsets of X will be denoted by T and pT, respectively.

Observe that a set $A \subset X$ is twin if and only if $\operatorname{Fix}^{-}(A)$ is not empty.

The notion of a twin set has an obvious "ideal" version.

For a left-invariant ideal $\mathcal{I}$ of subsets of a group X, and a subset $A \subset X$ consider the following subsets of X:

$$\mathcal{I}\text{-}\operatorname{Fix}(A) = \{x \in X : xA =_{\mathcal{I}} A\}, \quad \mathcal{I}\text{-}\operatorname{Fix}^{-}(A) = \{x \in X : xA =_{\mathcal{I}} X \setminus A\}, \quad \text{and } \mathcal{I}\text{-}\operatorname{Fix}^{\pm}(A) = \mathcal{I}\text{-}\operatorname{Fix}(A) \cup \mathcal{I}\text{-}\operatorname{Fix}^{-}(A).$$

Definition 5.2. A subset $A \subset X$ is defined to be
- *$\mathcal{I}$-twin* if $xA =_{\mathcal{I}} X \setminus A$ for some $x \in X$,
- *$\mathcal{I}$-pretwin* if $xA \subset_{\mathcal{I}} X \setminus A \subset_{\mathcal{I}} yA$ for some points $x, y \in X$.

The families of $\mathcal{I}$-twin and $\mathcal{I}$-pretwin subsets of X will be denoted by $\mathsf{T}^{\mathcal{I}}$ and $\mathsf{pT}^{\mathcal{I}}$, respectively.

It is clear that $\mathsf{T}^{\{\emptyset\}} = \mathsf{T}$ and $\mathsf{pT}^{\{\emptyset\}} = \mathsf{pT}$.

Proposition 5.3. *For each subset $A \subset X$ the set $\mathcal{I}\text{-}\operatorname{Fix}^{\pm}(A)$ is a subgroup in X. The set A is $\mathcal{I}$-twin if and only if $\mathcal{I}\text{-}\operatorname{Fix}(A)$ is a normal subgroup of index 2 in $\mathcal{I}\text{-}\operatorname{Fix}^{\pm}(A)$.*

Proof. If the set A is not $\mathcal{I}$-twin, then $\mathcal{I}\text{-}\operatorname{Fix}^{-}(A) = \emptyset$ and then $\mathcal{I}\text{-}\operatorname{Fix}^{\pm}(A) = \mathcal{I}\text{-}\operatorname{Fix}(A) = \{x \in X : xA =_{\mathcal{I}} A\}$ is a subgroup of X by the transitivity and the left-invariance of the equivalence relation $=_{\mathcal{I}}$.

So, we assume that A is $\mathcal{I}$-twin, which means that $\mathcal{I}\text{-}\operatorname{Fix}^{-}(A) \neq \emptyset$. To show that $\mathcal{I}\text{-}\operatorname{Fix}^{\pm}(A)$ is a subgroup in X, take any two points $x, y \in \mathcal{I}\text{-}\operatorname{Fix}^{\pm}(A)$. We claim that $xy^{-1} \in \mathcal{I}\text{-}\operatorname{Fix}^{\pm}(A)$.

This is clear if $x, y \in \mathcal{I}\text{-}\operatorname{Fix}(A) \subset \mathcal{I}\text{-}\operatorname{Fix}^{\pm}(A)$. If $x \in \mathcal{I}\text{-}\operatorname{Fix}(A)$ and $y \in \mathcal{I}\text{-}\operatorname{Fix}^{-}(A)$, then $xA =_{\mathcal{I}} A$, $yA =_{\mathcal{I}} X \setminus A$ and thus $A =_{\mathcal{I}} X \setminus y^{-1}A$ which implies $y^{-1}A =_{\mathcal{I}} X \setminus A$. Then $xy^{-1}A =_{\mathcal{I}} x(X \setminus A) = X \setminus xA =_{\mathcal{I}} X \setminus A$, which means that $xy^{-1} \in \mathcal{I}\text{-}\operatorname{Fix}^{-}(A) \subset \operatorname{Fix}^{\pm}(A)$.

If $x, y \in \mathcal{I}\text{-}\operatorname{Fix}^{-}(A)$, then $xA =_{\mathcal{I}} X \setminus A$, $y^{-1}A =_{\mathcal{I}} X \setminus A$. This implies that $xy^{-1}A =_{\mathcal{I}} x(X \setminus A) =_{\mathcal{I}} X \setminus xA =_{\mathcal{I}} X \setminus (X \setminus A) =_{\mathcal{I}} A$ and consequently $xy^{-1} \in \mathcal{I}\text{-}\operatorname{Fix}(A)$.

To show that $\mathcal{I}$-$\mathrm{Fix}(A)$ is a subgroup of index 2 in $\mathrm{Fix}^{\pm}(A)$, fix any element $g \in \mathcal{I}$-$\mathrm{Fix}^-(A)$. Then for every $x \in \mathcal{I}$-$\mathrm{Fix}(A)$ we get $gxA =_{\mathcal{I}} gA =_{\mathcal{I}} X \setminus A$ and thus $gx \in \mathcal{I}$-$\mathrm{Fix}^-(A)$. This yields $\mathcal{I}$-$\mathrm{Fix}^-(A) = g(\mathcal{I}$-$\mathrm{Fix}A)$, which means that the subgroup $\mathcal{I}$-$\mathrm{Fix}(A)$ has index 2 in the group $\mathcal{I}$-$\mathrm{Fix}^{\pm}(A)$. $\qquad\square$

The following proposition shows that the family $\mathsf{T}^{\mathcal{I}}$ of $\mathcal{I}$-twin sets of a group X is left-invariant.

Proposition 5.4. *For any $\mathcal{I}$-twin set $A \subset X$ and any $x \in X$ the set xA is $\mathcal{I}$-twin and $\mathcal{I}$-$\mathrm{Fix}^-(xA) = x\left(\mathcal{I}\text{-}\mathrm{Fix}^-(A)\right)x^{-1}$.*

Proof. To see that xA is an $\mathcal{I}$-twin set, take any $z \in \mathcal{I}$-$\mathrm{Fix}^-(A)$ and observe that

$$X \setminus xA = x(X \setminus A) =_{\mathcal{I}} xzA = xzx^{-1}xA,$$

which means that $xzx^{-1} \in \mathcal{I}$-$\mathrm{Fix}^-(xA)$ for every $z \in \mathcal{I}$-$\mathrm{Fix}^-(A)$. Hence $\mathcal{I}$-$\mathrm{Fix}^-(xA) = x\left(\mathcal{I}\text{-}\mathrm{Fix}^-(A)\right)x^{-1}$. $\quad\square$

The preceding proposition implies that the family $\mathsf{T}^{\mathcal{I}}$ of $\mathcal{I}$-twin subsets of X can be considered as an X-act with respect to the left action

$$\cdot : X \times \mathsf{T}^{\mathcal{I}} \to \mathsf{T}^{\mathcal{I}}, \ \cdot : (x, A) \mapsto xA$$

of the group X. By $[A] = \{xA : x \in X\}$ we denote the orbit of a $\mathcal{I}$-twin set $A \in \mathsf{T}^{\mathcal{I}}$ and by $[\mathsf{T}^{\mathcal{I}}] = \{[A] : A \in \mathsf{T}^{\mathcal{I}}\}$ the orbit space. If $\mathcal{I} = \{\emptyset\}$ is a trivial ideal, then we write $[\mathsf{T}]$ instead of $[\mathsf{T}^{\mathcal{I}}]$.

6. Twinic groups

An left-invariant ideal $\mathcal{I}$ on a group X is called *twinic* if for any subset $A \subset X$ and points $x, y \in X$ with $xA \subset_{\mathcal{I}} X \setminus A \subset_{\mathcal{I}} yA$ we get $A =_{\mathcal{I}} B$. In this case the families $\mathsf{pT}^{\mathcal{I}}$ and $\mathsf{T}^{\mathcal{I}}$ coincide.

A group X is defined to be *twinic* if it admits a twinic ideal $\mathcal{I}$. It is clear that in a twinic group X the intersection $I\!\!I$ of all twinic ideals is the smallest twinic ideal in X called *the twinic ideal* of X. The structure of the twinic ideal $I\!\!I$ can be described as follows.

Let $I\!\!I_0 = \{\emptyset\}$ and for each $n \in \omega$ let $I\!\!I_{n+1}$ be the ideal generated by sets of the form $yA \setminus xA$ where $xA \subset_{I\!\!I_n} X \setminus A \subset_{I\!\!I_n} yA$ for some $A \subset X$ and $x, y \in X$. By induction it is easy to check that $I\!\!I_n \subset I\!\!I_{n+1} \subset \mathcal{I}$ and hence $I\!\!I = \bigcup_{n \in \omega} I\!\!I_n \subset \mathcal{I}$ is a well-defined (smallest) twinic ideal on X.

In fact, the above constructive definition of the family $I\!\!I$ is valid for each group X. However, $I\!\!I$ is an ideal if and only if the group X is twinic.

We shall say that a group X has *trivial twinic ideal* if the trivial ideal $\mathcal{I} = \{\emptyset\}$ is twinic. This happen if and only if for any subset $A \subset X$ with $xA \subset X \setminus A \subset yA$ we get $xA = X \setminus A = yA$. In this case the twinic ideal $I\!\!I$ of X is trivial.

The class of twinic groups is sufficiently wide and contains all amenable groups. Let us recall that a group X is called *amenable* if it admits a Banach measure $\mu : \mathsf{P}(X) \to [0, 1]$, which is a left-invariant probability measure defined on the family of all subsets of $\mathsf{P}(X)$. In this case the family

$$\mathcal{N}_\mu = \{A \subset X : \mu(A) = 0\}$$

is an left-invariant ideal in X. It is well-known that the class of amenable groups contains all abelian groups and is closed with respect to many operations over groups, see [15].

A subset A of amenable group X is called *absolutely null* if $\mu(A) = 0$ for each Banach measure μ on X. The family $\mathcal{N}$ of all absolutely null subsets is an ideal on X. This ideal coincides with the intersection $\mathcal{N} = \bigcap_\mu \mathcal{N}_\mu$ where μ runs over all Banach measure of X.

Theorem 6.1. *Each amenable group X is twinic. The twinic ideal $I\!\!I$ of X lies in the ideal $\mathcal{N}$ of absolute null subsets of X.*

Proof. It suffices to check that the ideal $\mathcal{N}$ is twinic. Take any set $A \subset X$ such that $xA \subset_{\mathcal{N}} X \setminus A \subset_{\mathcal{N}} yA$ for some $x, y \in X$. We need to show that $\mu(yA \setminus xA) = 0$ for each Banach measure μ on X. It follows from $xA \subset_{\mathcal{N}} X \setminus A \subset_{\mathcal{N}} yA$ and the invariance of the Banach measure μ that

$$\mu(A) = \mu(xA) \le \mu(X \setminus A) \le \mu(yA) = \mu(A)$$

and hence $\mu(yA \setminus xA) = \mu(A) - \mu(A) = 0$. $\qquad\square$

Next, we show that the class of twinic groups contains also some non-amenable groups. The simplest example is the Burnside group $B(n, m)$ for $n \geq 2$ and odd $m \geq 665$. We recall that the Burnside group $B(n, m)$ is generated by n elements and one relation $x^m = 1$. Adian [1] proved that for $n \geq 2$ and any odd $m \geq 665$ the Burnside group $B(n, m)$ is not amenable, see also [14] for a stronger version of this result. The following theorem implies that each Burnside group, being a torsion group, is twinic. Moreover, its twinic ideal $\mathit{I\!I}$ is trivial!

Theorem 6.2. *A group X has trivial twinic ideal $\mathit{I\!I} = \{\emptyset\}$ if and only if the product ab of any elements $a, b \in X$ belongs to the subsemigroup of X generated by the set $b^{\pm} \cdot a^{\pm}$ where $a^{\pm} = \{a, a^{-1}\}$.*

Proof. To prove the "only if" part, assume that $\mathit{I\!I} \neq \{\emptyset\}$. Then $\mathit{I\!I}_{n+1} \neq \{\emptyset\} = \mathit{I\!I}_n$ for some $n \in \omega$ and we can find a subset $A \subset X$ and points $a, b \in X$ such that $a^{-1}A \subset X \setminus A \subset bA$ but $aA \neq bA$. Consider the subsemigroup $\mathrm{Fix}_{\subset}(A) = \{x \in X : xA \subset A\} \subset X$ and observe that $b^{-1}a^{-1} \in \mathrm{Fix}_{\subset}(A)$. The inclusion $a^{-1}A \subset X \setminus A$ implies $a^{-1}A \cap A = \emptyset$ which is equivalent to $A \cap aA = \emptyset$ and yields $aA \subset X \setminus A \subset bA$. Then $b^{-1}a \in \mathrm{Fix}_{\subset}(A)$.

Now consider the chain of the equivalences

$$X \setminus A \subset bA \ \Leftrightarrow \ A \cup bA = X \ \Leftrightarrow \ b^{-1}A \cup A = X \ \Leftrightarrow \ X \setminus A \subset b^{-1}A$$

and combine the last inclusion with $aA \cup a^{-1}A \subset X \setminus A$ to obtain $ba, ba^{-1} \in \mathrm{Fix}_{\subset}(A)$. Now we see that the subsemigroup S of X generated by the set $\{1, ba, ba^{-1}, b^{-1}a, b^{-1}a^{-1}\}$ lies in $\mathrm{Fix}_{\subset}(A)$. Observe that $b^{-1}a^{-1}A \subsetneq A$ implies $abA \not\subset A$, $ab \notin \mathrm{Fix}_{\subset}(A) \supset S$, and finally $ab \notin S$. This completes the proof of the "only if" part.

To prove the "if" part, assume that the group X contains elements a, b whose product ab does not belong to the subsemigroup generated by $b^{\pm}a^{\pm}$ where $a^{\pm} = \{a, a^{-1}\}$ and $b^{\pm} = \{b, b^{-1}\}$. Then ab does not belongs also to the subsemigroup S generated by $\{1\} \cup b^{\pm}a^{\pm}$. Observe that $a^{\pm}S = S^{-1}a^{\pm}$ and $b^{\pm}S^{-1} = Sb^{\pm}$.

We claim that

(2) $$S \cap a^{\pm}S = \emptyset \text{ and } S \cap Sb^{\pm} = \emptyset.$$

Assuming that $S \cap a^{\pm}S \neq \emptyset$ we would find a point $s \in S$ such that $as \in S$ or $a^{-1}s \in S$. If $as \in S$, then $bs^{-1} = b(as)^{-1}a \in bS^{-1}a \subset Sb^{\pm}a \subset S$ and hence $b = bs^{-1}s \in S \cdot S \subset S$. Then $a^{\pm} = b(b^{-1}a^{\pm}) \subset S \cdot S \subset S$, $b^{\pm} = (b^{\pm}a)a^{-1} \in S \cdot S \subset S$ and finally $ab \in S \cdot S \subset S$, which contradicts $ab \notin S$. By analogy we can treat the case $a^{-1}s \in S$ and also prove that $S \cap Sb^{\pm} = \emptyset$.

Consider the family $\mathcal{P}$ of all pairs (A, B) of disjoint subsets of X such that

   (a) $a^{\pm}A \subset B$ and $b^{\pm}B \subset A$;
   (b) $S^{-1}B \subset B$;
   (c) $1 \in A$, $ab \in B$.

The family $\mathcal{P}$ is partially ordered by the relation $(A, B) \leq (A', B')$ defined by $A \subset A'$ and $B \subset B'$.

We claim that the pair $(A_0, B_0) = (S \cup Sb^{\pm}ab, S^{-1}a^{\pm} \cup S^{-1}ab)$ belongs to $\mathcal{P}$. Indeed,

$$a^{\pm}A_0 = a^{\pm}S \cup a^{\pm}Sb^{\pm}ab \subset S^{-1}a^{\pm} \cup S^{-1}a^{\pm}b^{\pm}ab \subset S^{-1}a^{\pm} \cup S^{-1}ab \subset B_0.$$

By analogy we check that $b^{\pm}B_0 \subset A_0$. The items (b), (c) trivially follow from the definition of A_0 and B_0. It remains to check that the sets A_0 and B_0 are disjoint.

This will follow as soon as we check that

   (d) $S \cap S^{-1}a^{\pm} = \emptyset$,
   (e) $S \cap S^{-1}ab = \emptyset$,
   (f) $Sb^{\pm}ab \cap S^{-1}a^{\pm} = \emptyset$,
   (g) $Sb^{\pm}ab \cap S^{-1}ab = \emptyset$.

The items (d) and (g) follow from (2). The item (e) follows from $ab \notin S \cdot S = S$. By the same reason, we get the item (f) which is equivalent to $ab \notin b^{\pm}S^{-1} \cdot S^{-1}a^{\pm} = b^{\pm}S^{-1}a^{\pm} = Sb^{\pm}a^{\pm} \subset S$.

Thus the partially ordered set $\mathcal{P}$ is not empty and we can apply the Zorn Lemma to find a maximal pair $(A, B) \geq (A_0, B_0)$ in $\mathcal{P}$. We claim that $A \cup B = X$. Assuming the converse, we could take any point $x \in X \setminus (A \cup B)$ and put $A' = A \cup Sx$, $B' = B \cup a^{\pm}Sx$. It is clear that $a^{\pm}A' \subset B'$ and $b^{\pm}B' \subset A'$, $S^{-1}B' = S^{-1}B \cup S^{-1}a^{\pm}Sx \subset B \cup a^{\pm}SSx = B'$, $1 \in A \subset A'$ and $ab \in B \subset B'$.

Now we see that the inclusion $(A', B') \in \mathcal{P}$ will follow as soon as we check that $A' \cap B' = \emptyset$. The choice of $x \notin B = S^{-1}B$ guarantees that $Sx \cap B = \emptyset$. Assuming that $a^{\pm}Sx \cap A \neq \emptyset$, we would conclude that $x \in S^{-1}a^{\pm}A \subset S^{-1}B \subset B$, which contradicts the choice of x. Finally, the sets Sx and $a^{\pm}Sx$ are disjoint because of the property (2) of S. Thus we obtain a contradiction: $(A', B') \in \mathcal{P}$ is strictly greater than the maximal pair (A, B). This contradiction shows that $X = A \cup B$ and consequently, $aA \subset X \setminus A \subset B \subset bA$, which means that the set A is pretwin and then $bA \setminus aA \in \mathcal{II}_1 \subset \mathcal{II}$. Since $1 \in A \setminus b^{-1}a^{-1}A$, we conclude that $bA \setminus a^{-1}A \ni b$ is not empty and thus $\mathcal{II} \neq \{\emptyset\}$. $\square$

We recall that a group X is *periodic* (or else a *torsion group*) if each element $x \in X$ has finite order (which means that $x^n = e$ for some $n \in \mathbb{N}$). We shall say that a group X has *periodic commutators* if for any $x, y \in G$ the commutator $[x, y] = xyx^{-1}y^{-1}$ has finite order in X. It is intersecting to note that this condition is strictly weaker than the requirement for X to have periodic commutator subgroup X' (we recall that the commutator subgroup X' coincides with the set of finite products of commutators), see [6].

Proposition 6.3. *Each group X with periodic commutators has trivial twinic ideal $\mathcal{II} = \{\emptyset\}$.*

Proof. Since X has periodic commutators, for any points $x, y \in X$ there is a number $n \in \mathbb{N}$ such that

$$xyx^{-1}y^{-1} = (yxy^{-1}x^{-1})^{-1} = (yxy^{-1}x^{-1})^n$$

and thus $xy = (yxy^{-1}x^{-1})^n \cdot yx$ belongs to the semigroup generated by the set $y^{\pm} \cdot x^{\pm}$. Applying Theorem 6.2, we conclude that the group X has trivial twinic ideal $\mathcal{II} = \{\emptyset\}$. $\square$

We recall that a group G is called *abelian-by-finite* (resp. *finite-by-abelian*) if G contains a normal Abelian (resp. finite) subgroup $H \subset G$ with finite (resp. Abelian) quotient G/H. Observe that finite-by-abelian groups has periodic commutators and hence has trivial twinic ideal $\mathcal{II}$.

In contrast, any abelian-by-finite groups, being amenable, is twinic but its twinic ideal $\mathcal{II}$ need not be trivial. The simplest counterexample is the isometry group $\mathrm{Iso}(\mathbb{Z})$ of the group $\mathbb{Z}$ of integers endowed with the Euclidean metric.

Example 6.4. The abelian-by-finite group $X = \mathrm{Iso}(\mathbb{Z})$ is twinic. Its twinic ideal $\mathcal{II}$ coincides with the ideal $[X]^{<\omega}$ of all finite subsets of X.

Proof. Let $a : x \mapsto x + 1$ be the translation and $b : x \mapsto -x$ be the inversion of the group $\mathbb{Z}$. It is easy to see that the elements a, b generate the isometry group $X = \mathrm{Iso}(\mathbb{Z})$ and satisfy the relations $b^2 = 1$ and $bab^{-1} = a^{-1}$. Let $Z = \{a^n : n \in \mathbb{Z}\}$ be the cyclic subgroup of X generated by the translation a. This subgroup Z has index 2 in $X = Z \cup Zb$.

First we show that the ideal $\mathcal{I} = [X]^{<\omega}$ of finite subsets of X is twinic. Let $A \subset X$ be a subset with $xA \subset_{\mathcal{I}} X \setminus A \subset_{\mathcal{I}} yA$ for some $x, y \in X$. We need to show that $yA =_{\mathcal{I}} xA$.

We consider three cases.

1) $x, y \in Z$. In this case the elements x, y commute. The $\mathcal{I}$-inclusion $xA \subset_{\mathcal{I}} yA$ implies $y^{-1}xA \subset_{\mathcal{I}} A$. We claim that $y^{-1}xA \supset_{\mathcal{I}} A$. Observe that the $\mathcal{I}$-inclusion $xA \subset_{\mathcal{I}} X \setminus A$ is equivalent to $xA \cap A \in \mathcal{I}$ and to $A \cap x^{-1}A \in \mathcal{I}$, which implies $x^{-1}A \subset_{\mathcal{I}} X \setminus A$. By analogy, $X \setminus A \subset_{\mathcal{I}} yA$ is equivalent to $yA \cup A =_{\mathcal{I}} X$ and to $A \cup y^{-1}A =_{\mathcal{I}} X$, which implies $X \setminus A \subset_{\mathcal{I}} y^{-1}A$. Then $x^{-1}A \subset_{\mathcal{I}} X \setminus A \subset_{\mathcal{I}} y^{-1}A$ implies $yx^{-1}A \subset_{\mathcal{I}} A$ and by the left-invariance of $\mathcal{I}$, $A \subset_{\mathcal{I}} xy^{-1}A = y^{-1}xA$ (we recall that the elements x, y^{-1} commute). Therefore, $y^{-1}xA =_{\mathcal{I}} A$ and hence $xA =_{\mathcal{I}} yA$.

2) $x \in Z$ and $y \in X \setminus Z$. Repeating the argument from the preceding case, we can check that $xA \subset_{\mathcal{I}} X \setminus A$ implies $x^{-1}A \subset_{\mathcal{I}} X \setminus A$. Then we get the chain of $\mathcal{I}$-inclusions:

$$xA \subset_{\mathcal{I}} X \setminus A \subset_{\mathcal{I}} yA \subset_{\mathcal{I}} y(X \setminus xA) = yx(X \setminus A) \subset_{\mathcal{I}} yxyA =_{\mathcal{I}} xA,$$

where the last $\mathcal{I}$-equality follows from the case (1) since $x, yxy \in Z$. Now we see that $xA =_{\mathcal{I}} yA$.

3) $x \notin Z$. Then $xA \subset_{\mathcal{I}} X \setminus A \subset_{\mathcal{I}} yA$ implies

$$x^{-1}bA = bxb^{-1}bA = bxA \subset_{\mathcal{I}} X \setminus bA \subset_{\mathcal{I}} byA = y^{-1}bA.$$

Since $x^{-1}b \in Z$, the cases (1),(2) imply the $\mathcal{I}$-equality $x^{-1}bA =_{\mathcal{I}} y^{-1}bA$. Shifting this equality by b, we see that $xA = bx^{-1}bA =_{\mathcal{I}} by^{-1}bA = yA$.

This completes the proof of the twinic property of the ideal $\mathcal{I} = [X]^{<\omega}$. Then the twinic ideal $\mathcal{II} \subset [X]^{<\omega}$. Since $[X]^{<\omega}$ is the smallest non-trivial ideal on X, the equality $\mathcal{II} = [X]^{<\omega}$ will follow as soon as we find a non-empty set in the ideal $\mathcal{II}$.

For this consider the subset $A = \{a^{n+1}, ba^{-n} : n \geq 0\} \subset X$ and observe that $X \setminus A = \{a^{-n}, ba^{n+1} : n \geq 0\} = bA$ witnessing that $A \in \mathsf{T}$. Observe also that $aA = \{a^{n+2}, aba^{-n} : n \geq 0\} = \{a^{n+2}, ba^{-n-1} : n \geq 0\} \subsetneq A$ and thus $baA \subsetneq X \setminus A = bA$. Then $\emptyset \neq baA \setminus bA \in \mathcal{II}_1 \subset \mathcal{II}$ witnesses that the twinic ideal $\mathcal{II}$ is not trivial. $\square$

Next, we present (an expected) example of a group, which is not twinic.

Example 6.5. The free group F_2 with two generators is not twinic.

Proof. Assume that the group $X = F_2$ is twinic and let $\mathcal{II}$ be the twinic ideal of F_2. Let a, b be the generators of the free group F_2. Each element $w \in F_2$ can be represented by a word in the alphabet $\{a, a^{-1}, b, b^{-1}\}$. The word of the smallest length representing w is called the *irreducible representation* of w. The irreducible word representing the neutral element of F_2 is the empty word. Let A (resp. B) be the set of words whose irreducible representation start with letter a or a^{-1} (resp. b or b^{-1}). Consider the subset

$$C = \{a^{2n}w : w \in B \cup \{e\}, \ n \in \mathbb{Z}\} \subset F_2$$

and observe that $abaC \subset X \setminus C = aC$. Then $aC \setminus abaC \in \mathcal{II}_1$ by the definition of the subideal $\mathcal{II}_1 \subset \mathcal{II}$. Observe that $a^3baC \subset aC \setminus abaC$ and thus $a^3baC \in \mathcal{II}_1$. Then also $C \in \mathcal{II}_1$ and $X \setminus C = aC \in \mathcal{II}_1$ by the left-invariance of $\mathcal{II}_1$. By the additivity of $\mathcal{II}_1$, we finally get $X = C \cup (X \setminus C) \in \mathcal{II}_1 \subset \mathcal{II}$, which is a desired contradiction. $\square$

Next, we prove some permanence properties of the class of twinic groups.

Proposition 6.6. *Let $f : X \to Y$ be a surjective group homomorphism. If the group X is twinic, then so is the group Y.*

Proof. Let $\mathcal{II}_X$ be the twinic ideal of X. It is easy to see that $\mathcal{I} = \{B \subset Y : f^{-1}(B) \in \mathcal{II}_X\}$ is a left-invariant ideal on the group Y. We claim that it is twinic. Given any subset $A \subset Y$ with $xA \subset_{\mathcal{I}} Y \setminus A \subset_{\mathcal{I}} yA$ for some $x, y \in Y$, let $B = f^{-1}(A)$ and observe that $x'B \subset_{\mathcal{II}} X \setminus B \subset_{\mathcal{II}} y'B$ for some points $x' \in f^{-1}(x)$ and $y' \in f^{-1}(y)$. The twinic property of the twinic ideal $\mathcal{II}$ guarantees that $f^{-1}(yA \setminus xA) = y'B \setminus x'B \in \mathcal{II}$, which implies $yA \setminus xA \in \mathcal{I}$ and hence $xA =_{\mathcal{I}} Y \setminus A =_{\mathcal{I}} yA$. $\square$

Problem 6.7. Is a subgroup of a twinic group twinic? Is the product of two twinic groups twinic?

For groups with trivial twinic ideal the first part of this problem has an affirmative solution, which follows from the characterization Theorem 6.2.

Proposition 6.8.   (1) *The class of groups with trivial twinic ideal is closed with respect to taking subgroups and quotient groups.*
   (2) *A group X has trivial twinic ideal if and if any 2-generated subgroup of X has trivial twinic ideal.*

7. 2-Cogroups

It follows from Proposition **??** that for a twin subset $A \subset X$ the stabilizer $\mathrm{Fix}(A)$ of A is completely determined by the subset $\mathrm{Fix}^-(A)$ because $\mathrm{Fix}(A) = x \cdot \mathrm{Fix}^-(A)$ for each $x \in \mathrm{Fix}^-(A)$. Therefore, the subset $\mathrm{Fix}^-(A)$ carries all the information about the pair $(\mathrm{Fix}^\pm(A), \mathrm{Fix}(A))$. The sets $\mathrm{Fix}^-(A)$ are particular cases of so-called 2-cogroups defined as follows.

Definition 7.1. A subset K of a group X is called a *2-cogroup* if for every $x \in K$ the shift $xK = Kx$ is a subgroup of X, disjoint with K.
   By the *index* of a 2-cogroup K in X we understand the cardinality of the set $X/K = \{xK : x \in X\}$.

2-Cogroups can be characterized as follows.

Proposition 7.2. *A subset K of a group X is a 2-cogroup in X if and only if there is a (unique) subgroup $H^\pm$ of X and a subgroup $H \subset H^\pm$ of index 2 such that $K = H^\pm \setminus H$ and $H = K \cdot K$.*

Proof. If K is a 2-cogroup, then for every $x \in K$ the shift $H = xK = Kx$ is a subgroup of X disjoint with K. It follows that $K = x^{-1}H = Hx^{-1}$. Since $x^{-1} \in x^{-1}H = K$, the shift $x^{-1}K = Kx^{-1}$ is a group. Consequently, $x^{-1}Kx^{-1}K = x^{-1}K$, which implies $Kx^{-1}K = K$ and $Hx^{-1}x^{-1}Hx^{-1} = Kx^{-1}K = K = Hx^{-1}$. This implies $x^{-2} \in H$ and $x^2 \in H$. Consequently, $xH = x^{-1}x^2H = x^{-1}H = K = Hx^{-1} = Hx^2x^{-1} = Hx$.

Now we are able to show that $H^{\pm} = H \cup K$ is a group. Indeed,

$$(H \cup K) \cdot (H \cup K)^{-1} \subset HH^{-1} \cup HK^{-1} \cup KH^{-1} \cup KK^{-1} \subset$$

$$\subset H \cup HHx \cup xHH \cup Hx^{-1}xH = H \cup K \cup K \cup H = H^{\pm}.$$

Since $K = Hx = xH$, the subgroup $H = K \cdot K$ has index 2 in $H^{\pm}$. The uniqueness of the pair $(H^{\pm}, H)$ follows from the fact that $H = K \cdot K$ and $H^{\pm} = KK \cup K$. This completes the proof of the "only if" part.

To prove the "if" part, assume that $H^{\pm}$ is a subgroup of X and $H \subset H^{\pm}$ is a subgroup of index 2 such that $K = H^{\pm} \setminus H$. Then for every $x \in K$ the shift $xK = Kx = H$ is a subgroup of X disjoint with K. This means that K is a 2-cogroup. $\square$

By $\mathcal{K}$ we shall denote the family of all 2-cogroups in X. It is partially ordered by the inclusion relation $\subset$ and is considered as an X-act endowed with the conjugating action

$$\cdot : X \times \mathcal{K} \to \mathcal{K}, \quad \cdot : (x, K) \mapsto xKx^{-1},$$

of the group X. For each 2-cogroup $K \in \mathcal{K}$ let $\mathrm{Stab}(K) = \{x \in X : xKx^{-1} = K\}$ be the stabilizer of K and $[K] = \{xKx^{-1} : x \in X\}$ be the orbit of K. By $[\mathcal{K}] = \{[K] : K \in \mathcal{K}\}$ be denote the orbit space of $\mathcal{K}$ by the action of the group X.

Observe that for each 2-cogroup $K \in \mathcal{K}$ the stabilizer $\mathrm{Stab}(K)$ contains KK as a normal subgroup. So, we can consider the quotient group $H(K) = \mathrm{Stab}(K)/KK$ called the *characteristic group* of the 2-cogroup K. Characteristic groups will play an important role for understanding the structure of maximal subgroups of the minimal ideal of the semigroup $\lambda(X)$.

Since for each twin subset $A \subset X$ the set $\mathrm{Fix}^-(A)$ is a 2-cogroup, the function

$$\mathrm{Fix}^- : \mathsf{T} \to \mathcal{K}, \quad \mathrm{Fix}^- : A \mapsto \mathrm{Fix}^-(A),$$

is well-defined and equivariant according to Proposition **??**. A similar equivariant function

$$\mathcal{I}\text{-}\mathrm{Fix}^- : \mathsf{T}^{\mathcal{I}} \to \mathcal{K}, \quad \mathcal{I}\text{-}\mathrm{Fix}^- : A \mapsto \mathcal{I}\text{-}\mathrm{Fix}^-(A),$$

can be defined for any left-invariant ideal $\mathcal{I}$ on a group X.

Let $\widehat{\mathcal{K}}$ denote the set of maximal elements of the partially ordered set $(\mathcal{K}, \subset)$. The following proposition implies that the set $\widehat{\mathcal{K}}$ lies in the image $\mathrm{Fix}^-(\mathsf{T})$ and is cofinal in $\mathcal{K}$.

Proposition 7.3. (1) *For any linearly ordered family $\mathcal{C} \subset \mathcal{K}$ of 2-cogroups in X the union $\cup\mathcal{C}$ is a 2-cogroup in X.*
  (2) *Each 2-cogroup $K \in \mathcal{K}$ lies in a maximal 2-cogroup $\widehat{K} \in \widehat{\mathcal{K}}$.*
  (3) *For each maximal 2-cogroup $K \in \widehat{\mathcal{K}}$ there is a twin subset $A \in \mathsf{T}$ with $K = \mathrm{Fix}^-(A)$.*

Proof. 1. Let $\mathcal{C} \subset \mathcal{K}$ be a linearly ordered family of 2-cogroups of X. Since each 2-cogroup $C \in \mathcal{C}$ is disjoint with the group $C \cdot C$ and $C = C \cdot C \cdot C$, we get that the union $K = \cup\mathcal{C}$ is disjoint with the union $\bigcup_{C \in \mathcal{C}} C \cdot C = K \cdot K$ and $K = \bigcup_{C \in \mathcal{C}} C = \bigcup_{C \in \mathcal{C}} C \cdot C \cdot C = K \cdot K \cdot K$ witnessing that K is a 2-cogroup.

2. Since each chain in $\mathcal{K}$ is upper bounded, the Zorn Lemma guarantees that each 2-cogroup of X lies in a maximal 2-cogroup.

3. Given a maximal 2-cogroup $K \in \widehat{\mathcal{K}}$, consider the subgroups $H = K \cdot K$ and $H^{\pm} = K \cup H$ of X and choose a subset $S \subset G$ meeting each coset $H^{\pm}x$, $x \in X$, at a single point. Consider the set $A = H \cdot S$ and note that $X \setminus A = KS = xA$ for each $x \in K$, which means that $K \subset \mathrm{Fix}^-(A)$. The maximality of K guarantees that $K = \mathrm{Fix}^-(A)$. $\square$

It should be mentioned that in general, $\mathrm{Fix}^-(\mathsf{T}) \neq \mathcal{K}$.

Example 7.4. For any twin subset A in the 4-element group $X = C_2 \oplus C_2$ the group $\mathrm{Fix}(A)$ is not trivial. Consequently, each singleton $\{a\} \subset X \setminus \{e\}$ is a 2-cogroup that does not belong to the image $\mathrm{Fix}^-(\mathsf{T})$.

The family $\widehat{\mathcal{K}}$ of maximal 2-cogroup allows us to define an important family

$$\widehat{\mathsf{T}} = \{A \in \mathsf{T} : \mathrm{Fix}^-(A) \in \widehat{\mathcal{K}}\}$$

of twin sets. It is clear that

$$\widehat{\mathsf{T}} = \bigcup_{K \in \widehat{\mathcal{K}}} \mathsf{T}_K = \bigcup_{[K] \in [\widehat{\mathcal{K}}]} \mathsf{T}_{[K]}$$

where

$$\mathsf{T}_K = \{A \in \mathsf{T} : \mathrm{Fix}^-(A) = K\} \text{ and } \mathsf{T}_{[K]} = \{A \in \mathsf{T} : \mathrm{Fix}^-(A) \in [K]\}$$

for a 2-cogroup $K \in \mathcal{K}$.

A left-invariant subfamily $\mathsf{F} \subset \widehat{\mathsf{T}}$ is called

- $\widehat{\mathcal{K}}$-*covering* if for each maximal 2-cogroup $K \in \widehat{\mathcal{K}}$ there is a twin set $A \in \mathsf{F}$ with $\mathrm{Fix}^-(A) = K$;
- *minimal* $\widehat{\mathcal{K}}$-*covering* if F coincides with each $\widehat{\mathcal{K}}$-covering left-invariant subfamily $\mathsf{E} \subset \mathsf{F}$.

Proposition 7.5. *For any function $f \in \mathrm{End}_\lambda(\mathsf{P}(X))$ the family $f(\widehat{\mathsf{T}})$ is left-invariant and $\widehat{\mathcal{K}}$-covering.*

Proof. The equivariance of the function f and the left-invariance of the family $\widehat{\mathsf{T}}$ imply the left-invariance of the family $f(\widehat{\mathsf{T}})$. To see that $f(\widehat{\mathsf{T}})$ is $\widehat{\mathcal{K}}$-covering, fix any maximal 2-cogroup $K \in \widehat{\mathcal{K}}$ and using Proposition 7.3, find a twin set $A \subset X$ with $\mathrm{Fix}^-(A) = K$. We claim that $\mathrm{Fix}^-(f(A)) = \mathrm{Fix}^-(A) = K$. By Corollary 4.4, the function f is equivariant and symmetric. Then for every $x \in \mathrm{Fix}^-(A)$, applying f to the equality $xA = X \setminus A$, we get

$$x\,f(A) = f(xA) = f(X \setminus A) = X \setminus f(A),$$

which means that $x \in \mathrm{Fix}^-(f(A))$ and thus $\mathrm{Fix}^-(A) \subset \mathrm{Fix}^-(f(A))$. Now the maximality of the 2-cogroup $\mathrm{Fix}^-(A)$ guarantees that $\mathrm{Fix}^-(f(A)) = \mathrm{Fix}^-(A)$. $\qquad\square$

The following proposition describing the structure of minimal $\widehat{\mathcal{K}}$-covering left-invariant families can be easily derived from the definitions.

Proposition 7.6. *A left-invariant subfamily $\mathsf{F} \subset \widehat{\mathsf{T}}$ is minimal $\widehat{\mathcal{K}}$-covering if and only if for each $K \in \widehat{\mathcal{K}}$ there is a set $A \in \mathsf{F}$ such that $\mathsf{F} \cap \mathsf{T}_{[K]} = [A]$.*

8. (Maximally) $\mathcal{I}$-independent families

Let $\mathcal{I}$ be a left-invariant ideal on a group X. A family $\mathsf{F} \subset \mathsf{P}(X)$ is called

- $\mathcal{I}$-*independent* if $\forall A, B \in \mathsf{F}$ $(A \subset_{\mathcal{I}} B \Rightarrow A =_{\mathcal{I}} B)$;
- *maximally* $\mathcal{I}$-*independent* if $\forall A, B \in \mathsf{F}$ $(A \subset_{\mathcal{I}} B \Rightarrow A = B)$.

Proposition 8.1. *A left-invariant ideal $\mathcal{I}$ on a group X is twinic if and only if the family $\mathsf{pT}^{\mathcal{I}}$ of $\mathcal{I}$-pretwin sets is $\mathcal{I}$-independent.*

Proof. First assume that the family $\mathsf{pT}^{\mathcal{I}}$ is $\mathcal{I}$-independent. To show that the ideal $\mathcal{I}$ is twinic, take any subset $A \subset X$ with $xA \subset_{\mathcal{I}} X \setminus A \subset_{\mathcal{I}} yA$ for some $x, y \in X$. Then $A \in \mathsf{T}^{\mathcal{I}}$ and also $xA, yA \in \mathsf{T}^{\mathcal{I}}$. Since $xA \subset_{\mathcal{I}} yA$, the $\mathcal{I}$-independence of the family $\mathsf{T}^{\mathcal{I}}$ implies that $xA =_{\mathcal{I}} yA$ and then $xA =_{\mathcal{I}} X \setminus A =_{\mathcal{I}} yA$, which means that the ideal $\mathcal{I}$ is twinic.

Now assume conversely that $\mathcal{I}$ is twinic and take two $\mathcal{I}$-pretwin sets $A \subset_{\mathcal{I}} B$. Since the sets A, B are $\mathcal{I}$-pretwin, there are elements $x, y \in X$ such that $xB \subset_{\mathcal{I}} X \setminus B$ and $X \setminus A \subset_{\mathcal{I}} yA$. Taking into account that

$$xB \subset_{\mathcal{I}} X \setminus B \subset_{\mathcal{I}} X \setminus A \subset_{\mathcal{I}} yA \subset_{\mathcal{I}} yB,$$

and $\mathcal{I}$ is twinic, we conclude that $X \setminus B =_{\mathcal{I}} X \setminus A$ and hence $A =_{\mathcal{I}} B$. $\qquad\square$

Proposition 8.2. *For a left-invariant ideal $\mathcal{I}$ a subfamily $\mathsf{F} \subset \mathsf{pT}^{\mathcal{I}}$ is (maximally) $\mathcal{I}$-independent if for each $A \in \mathsf{F}$ the subgroup $\mathcal{I}$-$\mathrm{Fix}(A)$ (resp. $\mathrm{Fix}(A)$) has finite index in X.*

Proof. Assume that for each $A \in \mathsf{F}$ the subgroup $\mathcal{I}\text{-Fix}(A)$ has finite index in X. To show that the family F is $\mathcal{I}$-independent, take any subsets $A \subset_{\mathcal{I}} B$ in F. Since both sets are $\mathcal{I}$-pretwin, there are $x, y \in X$ such that $xB \subset_{\mathcal{I}} X \setminus B$ and $X \setminus A \subset yA$. Taking into account that $A \subset_{\mathcal{I}} B$ we conclude that $X \setminus B \subset_{\mathcal{I}} X \setminus A$ and thus

$$B \subset_{\mathcal{I}} x^{-1}(X \setminus B) \subset_{\mathcal{I}} x^{-1}(X \setminus A) \subset_{\mathcal{I}} x^{-1}yA \subset_{\mathcal{I}} x^{-1}yB.$$

By induction, the inclusion $B \subset_{\mathcal{I}} x^{-1}yB$ implies $B \subset_{\mathcal{I}} (x^{-1}y)^n B$ for all $n \in \mathbb{N}$. Since the subgroup $\mathcal{I}\text{-Fix}(B)$ has finite index in X, there is a number $n \in \mathbb{N}$ such that $(x^{-1}y)^n \in \mathcal{I}\text{-Fix}(B)$. For this number n we get

$$B \subset_{\mathcal{I}} x^{-1}(X \setminus B) \subset_{\mathcal{I}} (X \setminus A) \subset_{\mathcal{I}} x^{-1}yB \subset_{\mathcal{I}} (x^{-1}y)^n B =_{\mathcal{I}} B$$

and hence $x^{-1}(X \setminus B) =_{\mathcal{I}} x^{-1}(X \setminus A)$, which implies $A =_{\mathcal{I}} B$.

Now assume that for each set $A \in \mathsf{F}$ the group $\text{Fix}(A)$ has finite index in X. Then also the subgroup $\mathcal{I}\text{-Fix}(A) \supset \text{Fix}(A)$ has finite index in X and hence the family F is $\mathcal{I}$-independent by the preceding discussion. The total independence of F will follow as soon as we show that $A = B$ for any two sets $A =_{\mathcal{I}} B$ in F. Assume conversely that $A \neq B$. Without loss of generality, $A \not\subset B$ and hence there is a point $b \in B \setminus A$. It follows from our assumption that the subgroup $H = \text{Fix}(A) \cap \text{Fix}(B)$ has finite index in X and hence $bH \subset B \setminus A$. This and the $\mathcal{I}$-equality $A =_{\mathcal{I}} B$ imply $B \setminus A \in \mathcal{I}$ and $bH \in \mathcal{I}$. Since the subgroup H has finite index in X, there is a finite set $F \subset X$ such that $FH = X$. Then $X = \bigcup_{x \in F} xH \in \mathcal{I}$ by the left-invariance of the ideal, which is a contradiction. $\square$

Proposition 8.3. *A minimal $\widehat{\mathcal{K}}$-covering left-invariant subfamily $\mathsf{F} \subset \widehat{\mathsf{T}}$ is maximally $\mathcal{I}$-independent if and only if it is $\mathcal{I}$-independent.*

Proof. Assume that F is $\mathcal{I}$-independent and take any two sets $A \subset_{\mathcal{I}} B$ in F. Then $A =_{\mathcal{I}} B$ by the $\mathcal{I}$-independence of F. Since $\text{Fix}^-(A) \subset \mathcal{I}\text{-Fix}^-(A)$, the maximality of $\text{Fix}^-(A) \in \widehat{\mathcal{K}}$ guarantees that $\mathcal{I}\text{-Fix}^-(A) = \text{Fix}^-(A)$. By the same reason, $\mathcal{I}\text{-Fix}^-(B) = \text{Fix}^-(B)$. Then $A =_{\mathcal{I}} B$ implies $\text{Fix}^-(A) = \mathcal{I}\text{-Fix}^-(A) = \mathcal{I}\text{-Fix}^-(B) = \text{Fix}^-(B)$ and hence $A, B \in \mathsf{T}_{[K]}$ for the maximal 2-cogroup $K = \text{Fix}^-(A) = \text{Fix}^-(B)$. Since the intersection $\mathsf{F} \cap \mathsf{T}_{[K]}$ lies in some orbit, we get $[A] = [B]$ and hence $A = xB$ for some $x \in X$. It follows from $B =_{\mathcal{I}} A = xB$ that $x \in \mathcal{I}\text{-Fix}(B) = \text{Fix}(B)$ and thus $A = xB = B$. $\square$

9. The characteristic group $H(K)$ of a 2-cogroup K

In this section we study the algebraic structure of the characteristic group $H(K)$ of a maximal 2-cogroup $K \in \widehat{\mathcal{K}}$ of a group X. In particular, we show that each finite characteristic group $H(K)$ is isomorphic either to the cyclic 2-group C_{2^n} or to the group of generalized quaternions Q_{2^n}.

Here $C_{2^n} = \{z \in \mathbb{C} : z^{2^n} = 1\}$ stands for the cyclic group of order 2^n and

$$C_{2^\infty} = \bigcup_{n \in \omega} C_{2^n}$$

denotes the quasi-cyclic 2-group.

Let $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ be the quaternion group. It is a multiplicative subgroup of the quaternion algebra $\mathbb{H}$. The multiplicative subgroup of $\mathbb{H}$ generated by the union $C_{2^{n-1}} \cup Q_8$ for $n \leq \infty$ is denoted by Q_{2^n} and is referred to as the *group of generalized quaternions*, see [17]. According to this definition, $Q_{2^n} = Q_8$ for $n \leq 3$.

It should be mentioned that for $n \geq 3$ the generalized quaternion group Q_{2^n} has presentation

$$\langle x, y \mid x^2 = y^{2^{n-2}},\ x^4 = 1,\ xyx^{-1} = y^{-1} \rangle.$$

The groups C_{2^n} and Q_{2^n} are examples of locally finite 2-groups. We recall that a group G is called a *2-group* if the order of each element of G is a power of 2. A group G is *locally finite* if each finitely generated subgroup of G is finite.

Theorem 9.1. *Let $K \in \widehat{\mathcal{K}}$ be a maximal 2-cogroup in a group X. Then*
   (1) *$H(K)$ is a 2-group with a unique 2-element subgroup.*
   (2) *each finite subgroup of $H(K)$ is isomorphic to C_{2^n} or Q_{2^n} for some $n \in \mathbb{N}$;*
   (3) *each infinite abelian subgroup of $H(K)$ is isomorphic to C_{2^∞};*
   (4) *each infinite locally finite non-abelian subgroup of $H(K)$ is isomorphic to Q_{2^∞}.*

Proof. 1. Let $q : \mathrm{Stab}(K) \to H(K)$ be the quotient homomorphism. Take any element $x \in K$ and consider its image $d = q(x)$. Since $K = xKK$, the image $q(K) = \{d\}$ is a singleton. Taking into account that $x \notin KK$ and $x^2 \in KK$, we see that the element d has order 2 in $H(K)$. We claim that any other element a of order 2 in $H(K)$ is equal to d. Assume conversely that some element $a \neq d$ of $H(K)$ has order 2.

Let $C^{\pm}$ be the subgroup of $H(K)$ generated by the elements a, d and C be the cyclic subgroup generated by the product ad. We claim that $d \notin C$. Assuming conversely that $d \in C$, we conclude that $d = (ad)^n$ for some $n \in \mathbb{Z}$. Then $a = add = ad(ad)^n = (ad)^{n+1} \in C$ and consequently $a = d$ (because cyclic groups contain at most one element of order 2). It is clear that $C^{\pm} = C \cup dC$, which means that the subgroup C has index 2 in $C^{\pm}$.

Consider the subgroups $H^{\pm} = q^{-1}(C^{\pm})$, $H = q^{-1}(C)$ and observe that the 2-cogroup $H^{\pm} \setminus H$ is strictly larger than K, which contradicts $K \in \widehat{\mathcal{K}}$.

Since d is a unique element of order 2 in $H(K)$, the cyclic subgroup D generated by d is normal in $H(K)$. Consequently, for each non-trivial subgroup $G \subset H(K)$ the product $D \cdot G = G \cdot D$ is a subgroup in $H(K)$. Now we see that G must contain d. Otherwise, dG would be a 2-cogroup in $H(K)$ and its preimage $q^{-1}(dG)$ would be a 2-cogroup in X that contains the 2-cogroup K as a proper subset, which is impossible as K is a maximal 2-cogroup in X.

Therefore each non-trivial subgroup of $H(K)$ contains d. This implies that each element $x \in H(K)$ has finite order which a power of 2, witnessing that $H(K)$ is a 2-group with a single element of order 2.

2. The second item follows from the first one and Theorem 6.3.6 [17] saying that each finite 2-group with a unique element of order 2 is isomorphic to C_{2^n} or Q_{2^n} for some $n \in \mathbb{N}$.

3. Let G be an infinite abelian subgroup of $H(K)$. Being an abelian 2-group, G is locally finite and hence can be written as the union $G = \bigcup_{k \in \omega} G_k$ of increasing sequence of finite subgroups. By the preceding item, each subgroup G_k is cyclic of order 2^{n_k} for some n_k. Now it is seen that G is isomorphic to the quasicyclic group C_{2^∞}.

4. Let G be an infinite non-commutative locally finite subgroup of $H(K)$. Write $G = \bigcup_{n \in \omega} G_n$ as the union $G = \bigcup_{k \in \omega} G_k$ of an increasing sequence of non-commutative finite subgroups G_k or order $|G_k| \geq 16$. By the item 2, each subgroup G_k is isomorphic to the generalized quaternion group $Q_{2^{n_k}}$ for some $n_k \geq 4$. It follows that G_k has a unique abelian subgroup A_k of order $2^{n_k}/2$ and this subgroup is cyclic. Moreover, for each element $x \in G_k \setminus A_k$ and each $a \in A_k$ we get $xax^{-1} = a^{-1}$. It follows that $A_k \subset A_{k+1}$ and the union $A = \bigcup_{k \in \omega} A_k$ is isomorphic to the quasicyclic group C_{2^∞} and has index 2 in G. Also for each $x \in G \setminus A$ and $a \in A$ we get $xax^{-1} = a^{-1}$, witnessing that G is isomorphic to the group Q_{2^∞}. $\square$

10. Twin-generated topologies on groups

In this section we study so-called twin-generated topologies on groups. The information obtains in this section will be used in Section 15 for studying the topological structure of maximal subgroups of the minimal ideal of the superextension $\lambda(X)$.

Given a twin subset A of a group X consider the topology τ_A on X generated by the subbase consisting of the right shifts Ax, $x \in X$. In the following proposition by the weight of a topological space we understand the smallest cardinality of a sub-base of its topology.

Proposition 10.1.   (1) *The topology τ_A turns X into a right-topological group.*
   (2) *If $Ax = xA$ for all $x \in \mathrm{Fix}^-(A)$, then the topology τ_A is zero-dimensional.*
   (3) *The topology τ_A is T_1 if and only if the intersection $\bigcap_{x \in A} Ax^{-1}$ is a singleton.*
   (4) *The weight of the space (X, τ_A) does not exceed the index of the subgroup $\mathrm{Fix}(A^{-1})$ in X.*

Proof. 1. It is clear that the topology τ_A is right-invariant.

2. If $Ax = xA$ for all $x \in \mathrm{Fix}^-(A)$, then the set $X \setminus A$ is open in the topology τ_A because $X \setminus A = xA = Ax$ for any $x \in \mathrm{Fix}^-(A)$. Consequently, the space (X, τ_A) has a base consisting of open-and-closed subsets, which means that it is zero-dimensional.

3. If the topology τ_A is T_1, then the intersection $\bigcap_{g \in A} Ag^{-1}$ of all open neighborhoods of the neutral element e of X consists of a single point e. Assuming conversely that $\bigcap_{g \in A} Ag^{-1}$ is a singleton $\{e\}$, for any two distinct points $x, y \in X$ we can find a shift Ag^{-1}, $g \in A$, that contains the neutral element e but not

yx^{-1}. Then the shift $Ag^{-1}x$ is an open subset of (X, τ) that contains x but not y, witnessing that the space (X, τ_A) is T_1.

4. To estimate the weight of the space (X, τ_A), choose a subset $S \subset X$ meeting each coset $x\mathrm{Fix}(A^{-1})$, $x \in X$, at a single point (here $\mathrm{Fix}(A^{-1}) = \{x \in X : xA^{-1} = A^{-1}\}$). Then the set S^{-1} meets each coset $\mathrm{Fix}(A^{-1})x$, $x \in X$, at a single point. It is easy to see that the family $\{Ax : x \in S^{-1}\}$ forms a sub-base of the topology of τ and hence the weight of (X, τ) does not exceed $X/\mathrm{Fix}(A^{-1})$. $\qquad\square$

A topology τ on a group X will be called *twin-generated* if τ is equal to the topology τ_A generated by some twin subset $A \subset X$, i.e., τ is generated by the sub-base $\{Ax : x \in X\}$.

Because of Theorem 9.1, we shall be especially interested in twin-generated topologies on the quasi-cyclic group C_{2^∞} and the infinite quaternion group Q_{2^∞}. First we consider two simple examples.

Example 10.2. (1) The Euclidean topology τ_E on $\mathcal{C}_{2^\infty}$ is generated by the twin subset $E = C_{2^\infty} \cap \{e^{i\varphi} : -\frac{\pi}{3} < \varphi < \frac{2\pi}{3}\}$;

(2) The Sorgefrey topology on C_{2^∞} is generated by the twin subset $A = C_{2^\infty} \cap \{e^{i\varphi} : 0 \le \varphi < \pi\}$.

In the following proposition by τ_E we denote the Euclidean topology on C_{2^∞}.

Proposition 10.3. *Each metrizable shift-invariant topology $\tau \supset \tau_E$ on the quasi-cyclic group C_{2^∞} is twin-generated.*

Proof. Let $E_0 = C_{2^\infty} \cap \{e^{i\varphi} : -\pi/3 < \varphi < 2\pi/3\}$ be the twin subset generating the Euclidean topology on C_{2^∞} and $E_n = C_{2^\infty} \cap \{e^{i\varphi} : |\varphi| < 3^{-n-1}\pi\}$ for $n \ge 1$. For every $n \in \mathbb{N}$ let $\varphi_n = \sum_{k=1}^n \pi/4^n$ and observe that $\varphi_\infty = \sum_{k=1}^\infty \pi/4^n = \pi/3$.

The metrizable space (C_{2^∞}, τ) is countable and hence zero-dimensional. Since $\tau \supset \tau_E$, there exists a neighborhood base $\{U_n\}_{n=1}^\infty \subset \tau$ at the unit 1 such that each set U_n is closed and open in τ and $U_n \subset E_n$ for all $n \in \mathbb{N}$.

The interested reader can check that the twin subset

$$A = (E_0 \setminus \bigcup_{n=1}^\infty e^{i\varphi_n} E_n) \cup \bigcup_{n=1}^\infty e^{i\varphi_n} U_n \cup \bigcup_{n=1}^\infty e^{i(\pi+\varphi_n)} E_n \setminus U_n$$

generates the topology τ. $\qquad\square$

Problem 10.4. Is each metrizable shift-invariant topology on C_{2^∞} twin-generated?

11. THE CHARACTERISTIC GROUP $H(A)$ OF A TWIN SUBSET A

In this section, given a twin subset $A \in \mathsf{T}$ of a group X we introduce a twin-generated topology on the characteristic group $H(K)$ of the 2-cogroup $K = \mathrm{Fix}^-(A)$.

Consider the intersection $B = A \cap \mathrm{Stab}(K) = B \cdot KK$ and the image $A' = q_A(B)$ of the set B under the quotient homomorphism $q_A : \mathrm{Stab}(K) \to H(K) = \mathrm{Stab}(K)/KK$. We claim that A' is a twin subset of $H(K)$.

Indeed, for every $x \in \mathrm{Fix}^-(A) = K \subset \mathrm{Stab}(K)$ we get $X \setminus A = xA$ and consequently, $\mathrm{Stab}(K) \setminus B = xB$ and $H(A) \setminus A' = zA'$ where $z \in q_A(x)$.

Now it is legal to endow the group $H(K)$ with the topology $\tau_{A'}$ generated by the twin subset A', that is generated by the sub-base $\{A'x : x \in H(K)\}$. By Proposition 10.1 the topology $\tau_{A'}$ turns the characteristic group $H(K)$ into a right-topological group, which will be called the *characteristic group* of A and will be denoted by $H(A)$. By Proposition 10.1, the characteristic group $H(A)$ is a T_1-space and its weight does not exceed the cardinality of $H(A)$.

The reader should be conscious of the fact that for two twin subsets $A, B \in \mathsf{T}$ with $\mathrm{Fix}^-(A) = \mathrm{Fix}^-(B)$ the characteristic group $H(A)$ and $H(B)$ are algebraically isomorphic but topologically they can be distinct, see Example 10.2.

12. CONSTRUCTING NICE IDEMPOTENTS IN THE SEMIGROUP $\mathrm{End}_\lambda(\mathsf{P}(X))$

In this section we prove the existence some special idempotents in the semigroup $\mathrm{End}_\lambda(\mathsf{P}(X))$. These idempotents will help us to describe the structure of the minimal ideal of the semigroup $\mathrm{End}_\lambda(X)$ and $\lambda(X)$ in Theorems 13.1 and Corollary 13.2.

In this section we assume that $\mathcal{I}$ is a left-invariant ideal in a group X. We recall that $\mathsf{pT}^{\mathcal{I}}$ and $\mathsf{T}^{\mathcal{I}}$ denote the families of $\mathcal{I}$-pretwin and $\mathcal{I}$-twin subsets of X. A function $f : \mathsf{F} \to \mathsf{P}(X)$ defined on a subfamily $\mathsf{F} \subset \mathsf{P}(X)$ is called $\mathcal{I}$-*free* if $f(A) = f(B)$ for any sets $A =_{\mathcal{I}} B$ in F.

Proposition 12.1. *There is an idempotent $e_{\mathcal{I}} \in \mathrm{End}_\lambda(\mathsf{P}(X))$ such that*
- *$e_{\mathcal{I}}(\mathsf{P}(X) \setminus \mathsf{pT}^{\mathcal{I}}) \subset \{\emptyset, X\}$;*
- *$e_{\mathcal{I}}|\mathsf{pT}^{\mathcal{I}} = \mathrm{id}|\mathsf{pT}^{\mathcal{I}}$;*
- *the function $e_{\mathcal{I}}$ restricted to $\mathsf{P}(X) \setminus \mathsf{pT}^{\mathcal{I}}$ is $\mathcal{I}$-free.*

Proof. Consider the family $\overset{\leftrightarrow}{N}{}^{\mathcal{I}}_2(X) \subset \mathsf{P}^2(X)$ of invariant $\mathcal{I}$-free linked systems on X, partially ordered by the inclusion relation. This set is not empty because it contains the invariant $\mathcal{I}$-free linked system $\{X \setminus A : A \in \mathcal{I}\}$. By the Zorn Lemma, the partially ordered set $\overset{\leftrightarrow}{N}{}^{\mathcal{I}}_2(X)$ contains a maximal element $\mathcal{L}$, which is a maximal invariant $\mathcal{I}$-free linked system on the group X. By the maximality, the system $\mathcal{L}$ is monotone. Now consider the family

$$\mathcal{L}^\perp = \{A \subset X : \forall L \in \mathcal{L} \ (A \cap L \neq \emptyset)\}.$$

Claim 12.2. $\mathcal{L}^\perp \setminus \mathcal{L} \subset \mathsf{pT}^{\mathcal{I}}$.

Proof. Fix any set $A \in \mathcal{L}^\perp \setminus \mathcal{L}$. First we check that $xA \cap A \in \mathcal{I}$ for some $x \in X$. Assuming the converse, we would conclude that the family $\mathcal{A} = \{A' \subset X : \exists x \in X \ (A' =_{\mathcal{I}} xA)\}$ is invariant, $\mathcal{I}$-free and linked, and so is the union $\mathcal{A} \cup \mathcal{L}$, which is not possible by the maximality of $\mathcal{L}$. So, there is $x \in X$ with $xA \cap A \in \mathcal{I}$, which is equivalent to $xA \subset_{\mathcal{I}} X \setminus A$.

Next, we find $y \in X$ such that $A \cup yA =_{\mathcal{I}} X$, which is equivalent to $X \setminus A \subset_{\mathcal{I}} yA$. Assuming that no such a point y exists, we conclude that for any $x, y \in X$ the union $xA \cup yA \neq_{\mathcal{I}} X$. Then $(X \setminus xA) \cap (X \setminus yA) = X \setminus (xA \cup yA) \notin \mathcal{I}$, which means that the family $\mathcal{B} = \{B \subset X : \exists x \in X \ (B =_{\mathcal{I}} X \setminus xA)\}$ is invariant $\mathcal{I}$-free and linked. We claim that $X \setminus A \in \mathcal{L}^\perp$. Assuming the converse, we would conclude that $X \setminus A$ misses some set $L \in \mathcal{L}$. Then $L \subset A$ and hence $A \in \mathcal{L}$ which is not the case. Thus $X \setminus A \in \mathcal{L}^\perp$. Since $\mathcal{L}$ is invariant and $\mathcal{I}$-free, $\mathcal{B} \subset \mathcal{L}^\perp$ and consequently, the union $\mathcal{B} \cup \mathcal{L}$, being an invariant $\mathcal{I}$-free linked system, coincides with $\mathcal{L}$. Then $X \setminus A \in \mathcal{L}$, which contradicts $A \in \mathcal{L}^\perp$. This contradiction shows that $X \setminus A \subset_{\mathcal{I}} yA$ for some $y \in X$.

Since $xA \subset_{\mathcal{I}} X \setminus A \subset_{\mathcal{I}} yA$, the set A is $\mathcal{I}$-pretwin. $\square$

Consider the function representation $\Phi_{\mathcal{L}} : \mathsf{P}(X) \to \mathsf{P}(X)$ of $\mathcal{L}$. By Propositions 4.3 and 4.5, the function $\Phi_{\mathcal{L}}$ is equivariant, monotone, $\mathcal{I}$-free, and $\Phi_{\mathcal{L}}(\mathsf{P}(X)) \subset \{\emptyset, X\}$.

It is clear that the function $e_{\mathcal{I}} : \mathsf{P}(X) \to \mathsf{P}(X)$ defined by

$$e_{\mathcal{I}}(A) = \begin{cases} A & \text{if } A \in \mathsf{pT}^{\mathcal{I}}, \\ \Phi_{\mathcal{L}}(A) & \text{otherwise} \end{cases}$$

has properties (1)–(3) of Proposition 12.1. It is also clear that $e_{\mathcal{I}} = e_{\mathcal{I}} \circ e_{\mathcal{I}}$ is an idempotent.

We claim that $e_{\mathcal{I}} \in \mathrm{End}_\lambda(\mathsf{P}(X))$. By Corollary 4.4, we need to check that $e_{\mathcal{I}}$ is equivariant, monotone and symmetric. The equivariance of $e_{\mathcal{I}}$ follows is the equivariance of the maps $\Phi_{\mathcal{L}}$ and id.

To show that $e_{\mathcal{I}}$ is monotone, take any two subsets $A \subset B$ of X and consider four cases.

1) If $A, B \notin \mathsf{pT}^{\mathcal{I}}$, then $e_{\mathcal{I}}(A) = \Phi_{\mathcal{L}}(A) \subset \Phi_{\mathcal{L}}(B) = e_{\mathcal{I}}(B)$ by the monotonicity of the function representation $\Phi_{\mathcal{L}}$ of the monotone family $\mathcal{L}$.

2) If $A, B \in \mathsf{pT}^{\mathcal{I}}$, then $e_{\mathcal{I}}(A) = A \subset B = e_{\mathcal{I}}(B)$.

3) $A \in \mathsf{pT}^{\mathcal{I}}$ and $B \notin \mathsf{pT}^{\mathcal{I}}$. We claim that $B \in \mathcal{L}$. Assuming that $B \notin \mathcal{L}$ and applying Claim 12.2, we get $B \notin \mathcal{L}^\perp$. Then B does not intersect some set $L \in \mathcal{L}$ and then $A \cap L = \emptyset$. It follows that the set $X \setminus A \supset L$ belongs to the maximal invariant $\mathcal{I}$-free linked system and so does the set $yA \supset_{\mathcal{I}} X \setminus A$ for some $y \in X$ (which exists as $A \in \mathsf{pT}^{\mathcal{I}}$). By the left-invariance of $\mathcal{L}$, we get $A \in \mathcal{L}$ which contradicts $X \setminus A \in \mathcal{L}$ and the linkedness property of $\mathcal{L}$. This contradiction proves that $B \in \mathcal{L}$. In this case $e_{\mathcal{I}}(A) = A \subset X = \Phi_{\mathcal{L}}(B) = e_{\mathcal{I}}(B)$.

4) $A \notin \mathsf{pT}^{\mathcal{I}}$ and $B \in \mathsf{pT}^{\mathcal{I}}$. In this case we prove that $A \notin \mathcal{L}$. Assuming conversely that $A \in \mathcal{L}$, we get $B \in \mathcal{L}$. Since $B \in \mathsf{pT}^{\mathcal{I}}$, there is a point $x \in X$ with $xB \subset_{\mathcal{I}} X \setminus B$. Since $\mathcal{L}$ is left-invariant, monotone and $\mathcal{I}$-free, we conclude that $X \setminus B \in \mathcal{L}$ which contradicts $B \in \mathcal{L}$. Thus $A \notin \mathcal{L}$ and $e_{\mathcal{I}}(A) = \Phi_{\mathcal{L}}(A) = \emptyset \subset e_{\mathcal{I}}(B)$.

Finally, we show that the function $e_\mathcal{I}$ is symmetric. If $A \in \mathsf{pT}^\mathcal{I}$, then $X \setminus A \in \mathsf{pT}^\mathcal{I}$ and then $e_\mathcal{I}(X \setminus A) = X \setminus A = X \setminus e_\mathcal{I}(A)$.

Next, assume that $A \notin \mathsf{pT}^\mathcal{I}$. If $A \in \mathcal{L}$, then $X \setminus A \notin \mathcal{L}$ by the linkedness of $\mathcal{L}$. In this case $e_\mathcal{I}(X \setminus A) = \emptyset = X \setminus X = X \setminus e_\mathcal{I}(A)$.

If $A \notin \mathcal{L}$, then by Claim 12.2, $A \notin \mathcal{L}^\perp$ and thus A is disjoint with some set $L \in \mathcal{L}$, which implies that $X \setminus A \in \mathcal{L}$. Then $e_\mathcal{I}(X \setminus A) = \Phi_\mathcal{L}(X \setminus A) = X = X \setminus \emptyset = X \setminus \Phi_\mathcal{L}(A) = X \setminus e_\mathcal{I}(A)$. $\qquad \square$

Our second special idempotent depends on a subfamily $\widetilde{\mathsf{T}}$ of the family
$$\widehat{\mathsf{T}} = \{A \in \mathsf{T} : \mathrm{Fix}^-(A) \in \widehat{\mathcal{K}}\}$$
of twin sets with maximal 2-cogroup.

Theorem 12.3. *If the ideal $\mathcal{I}$ is twinic, then for any maximally $\mathcal{I}$-independent left-invariant $\widehat{\mathcal{K}}$-covering subfamily $\widetilde{\mathsf{T}} \subset \widehat{\mathsf{T}}$ there is an idempotent $e_{\widetilde{\mathsf{T}}} \in \mathrm{End}_\lambda^\mathcal{I}(\mathsf{P}(X))$ such that*

(1) $e_{\widetilde{\mathsf{T}}}(\mathsf{P}(X) \setminus \mathsf{T}^\mathcal{I}) \subset \{\emptyset, X\}$;
(2) $e_{\widetilde{\mathsf{T}}}(\mathsf{T}^\mathcal{I}) \subset \widetilde{\mathsf{T}}$;
(3) $e_{\widetilde{\mathsf{T}}}|\{\emptyset, X\} \cup \widetilde{\mathsf{T}} = \mathrm{id}$.

Proof. The idempotent $e_{\widetilde{\mathsf{T}}}$ will be defined as the composition $e_{\widetilde{\mathsf{T}}} = \varphi \circ e_\mathcal{I}$ where $\varphi : \{\emptyset, X\} \cup \mathsf{T}^\mathcal{I} \to \{\emptyset, X\} \cup \widetilde{\mathsf{T}}$ is an equivariant $\mathcal{I}$-free function such that

(1) $\varphi \circ \varphi = \varphi$;
(2) $\varphi|\{\emptyset, X\} \cup \widetilde{\mathsf{T}} = \mathrm{id}$;
(3) $\varphi(\mathsf{T}^\mathcal{I}) \subset \widetilde{\mathsf{T}}$;
(4) $\mathcal{I}\text{-}\mathrm{Fix}^-(A) \subset \mathrm{Fix}^-(\varphi(A))$ for all $A \in \mathsf{T}^\mathcal{I}$.

To construct such a function φ, consider the family $\mathcal{F}$ of all possible functions $\varphi : D_\varphi \to \{\emptyset, X\} \cup \widetilde{\mathsf{T}}$ such that

- $\{\emptyset, X\} \cup \widetilde{\mathsf{T}} \subset D_\varphi \subset \{\emptyset, X\} \cup \mathsf{T}^\mathcal{I}$;
- the set D_φ is left-invariant;
- φ is equivariant and $\mathcal{I}$-free;
- $\varphi|\{\emptyset, X\} \cup \widetilde{\mathsf{T}} = \mathrm{id}$;
- $\mathcal{I}\text{-}\mathrm{Fix}^-(A) \subset \mathrm{Fix}^-(\varphi(A))$ for all $A \in D_\varphi$.

The family $\mathcal{F}$ is partially ordered by the relation $\varphi \leq \psi$ defined by $\psi|D_\varphi = \varphi$.

The set $\mathcal{F}$ is not empty because it contains the identity function id of $\{\emptyset, X\} \cup \widetilde{\mathsf{T}}$, which is $\mathcal{I}$-free because of the maximal $\mathcal{I}$-independence of the family $\widetilde{\mathsf{T}}$. By the Zorn Lemma, the family $\mathcal{F}$ contains a maximal element $\varphi : D_\varphi \to \{\emptyset, X\} \cup \widetilde{\mathsf{T}}$. We claim that $D_\varphi = \{\emptyset, X\} \cup \mathsf{T}^\mathcal{I}$. Assuming the converse, fix a set $A \in \mathsf{T}^\mathcal{I} \setminus D_\varphi$ and define a family $D_\psi = D_\varphi \cup \{xA : x \in X\}$. Next, we extend the function φ to a function $\psi : D_\psi \to \{\emptyset, X\} \cup \widetilde{\mathsf{T}}$.

We consider two cases.

1) Assume that $A =_\mathcal{I} B$ for some $B \in D_\varphi$. Then also $xA =_\mathcal{I} xB$ for all $x \in X$. In this case we define the function $\psi : D_\psi \to \{\emptyset, X\} \cup \widetilde{\mathsf{T}}$ assigning to each set $C \in D_\psi$ the set $\varphi(D)$ where $D \in D_\varphi$ is any set with $D =_\mathcal{I} C$. It can be shown that the function $\psi : D_\psi \to \{\emptyset, X\} \cup \widetilde{\mathsf{T}}$ belongs to the family $\mathcal{F}$, which contradicts the maximality of φ.

2) Assume that $A \neq_\mathcal{I} B$ for all $B \in D_\varphi$. By Proposition 7.3, the 2-cogroup $\mathcal{I}\text{-}\mathrm{Fix}^-(A)$ lies in a maximal 2-cogroup $K \in \widehat{\mathcal{K}}$. Since the $\widehat{\mathcal{K}}$-covering family $\widetilde{\mathsf{T}}$ meets the family $\mathsf{T}_{[K]}$, there is a twin set $B \in \widetilde{\mathsf{T}}$ such that $\mathrm{Fix}^-(B) = K$. In this case define the function $\psi : D_\psi \to \widehat{\mathsf{T}}$ by the formula

$$\psi(C) = \begin{cases} \varphi(C) & \text{if } C \in D_\varphi; \\ xB & \text{if } C = xA \text{ for some } x \in X. \end{cases}$$

If $xA =_\mathcal{I} yA$ for some $x, y \in X$, then $y^{-1}x \in \mathcal{I}\text{-}\mathrm{Fix}^-(A) \subset K = \mathrm{Fix}^-(B)$ and thus $xB = yB$, which means that the function ψ is well-defined and $\mathcal{I}$-free. Also it is clear that ψ is equivariant and hence belongs to the family $\mathcal{F}$, which is forbidden by the maximality of φ.

Thus the maximal function φ is defined on $D_\varphi = \mathsf{T}^{\mathcal{I}}$ and we can put $e_{\widetilde{\mathsf{T}}} = \psi \circ e_{\mathcal{I}}$ where $e_{\mathcal{I}} : \mathsf{P}(X) \to \{\emptyset, X\} \cup \mathsf{p}\mathsf{T}^{\mathcal{I}} = \{\emptyset, X\} \cup \mathsf{T}^{\mathcal{I}}$ is the idempotent constructed in Proposition 12.1. It follows from the properties of the functions φ and $e_{\mathcal{I}}$ that the function $e_{\widetilde{\mathsf{T}}}$ is equivariant and $\mathcal{I}$-free. Since the ideal $\mathcal{I}$ is twinic, the family $\mathsf{T}^{\mathcal{I}} = \mathsf{p}\mathsf{T}^{\mathcal{I}}$ is $\mathcal{I}$-independent and hence the monotonicity of the function φ follows automatically from its $\mathcal{I}$-free property. Then $e_{\widetilde{\mathsf{T}}}$ is monotone as the composition of two monotone functions. By Corollary 4.4, $e_{\widetilde{\mathsf{T}}} \in \mathrm{End}_\lambda(\mathsf{P}(X))$. $\qquad\square$

Theorem 12.3 and Proposition 8.3 imply:

Corollary 12.4. *If the ideal $\mathcal{I}$ is twinic, then for each minimal left-invariant $\widehat{\mathcal{K}}$-covering family $\widetilde{\mathsf{T}} \subset \widehat{\mathsf{T}}$ there is an idempotent $e_{\widetilde{\mathsf{T}}} \in \mathrm{End}_\lambda^{\mathcal{I}}(\mathsf{P}(X))$ such that*

(1) *$e_{\widetilde{\mathsf{T}}}(\mathsf{P}(X) \setminus \mathsf{T}^{\mathcal{I}}) \subset \{\emptyset, X\}$;*
(2) *$e_{\widetilde{\mathsf{T}}}(\mathsf{T}^{\mathcal{I}}) \subset \widetilde{\mathsf{T}}$;*
(3) *$e_{\widetilde{\mathsf{T}}}|\{\emptyset, X\} \cup \widetilde{\mathsf{T}} = \mathrm{id}$.*

13. THE MINIMAL IDEAL OF THE SEMIGROUPS $\lambda(X)$ AND $\mathrm{End}_\lambda(\mathsf{P}(X))$

In this section we apply Corollary 12.4 for describing the structure of the minimal ideals the semigroups $\lambda(X)$ and $\mathrm{End}_\lambda(\mathsf{P}(X))$.

Theorem 13.1. *For a twinic group X a function $f \in \mathrm{End}_\lambda(\mathsf{P}(X))$ belongs to the minimal ideal $K(\mathrm{End}_\lambda(\mathsf{P}(X))$ of the semigroup $\mathrm{End}_\lambda(P(X))$ if and only if the following two conditions hold:*

(1) *the family $f(\widehat{\mathsf{T}})$ is minimal $\widehat{\mathcal{K}}$-covering;*
(2) *$f(\mathsf{P}(X)) \subset \{\emptyset, X\} \cup f(\widehat{\mathsf{T}})$.*

Proof. Let $\widetilde{\mathsf{T}} \subset \widehat{\mathsf{T}}$ be a minimal $\widehat{\mathcal{K}}$-covering left-invariant family and $e_{\widetilde{\mathsf{T}}} \in \mathrm{End}_\lambda(\mathsf{P}(X))$ be an idempotent satisfying the conditions (1)–(3) of in Corollary 12.4. By Propositions 8.1 and 8.3, the family $\widetilde{\mathsf{T}}$ is maximally $\mathcal{I}$-independent for any twinic ideal $\mathcal{I}$ on X.

To prove the "if" part of the theorem, assume that f satisfies the conditions (1), (2). To show that f belongs to the minimal ideal $K(\mathrm{End}_\lambda(\mathsf{P}(X)))$, it suffices for each $g \in \mathrm{End}_\lambda(\mathsf{P}(X))$ to find $h \in \mathrm{End}_\lambda(\mathsf{P}(X))$ such that $h \circ g \circ f = f$.

The minimality and the left-invariance of the $\widehat{\mathcal{K}}$-covering subfamily $f(\widehat{\mathsf{T}})$ imply that the equivariant function $\psi = e_{\widetilde{\mathsf{T}}} \circ g|f(\widehat{\mathsf{T}}) : f(\widehat{\mathsf{T}}) \to \widetilde{\mathsf{T}}$ is bijective. So, we can consider the inverse function $\psi^{-1} : \{\emptyset, X\} \cup \widetilde{\mathsf{T}} \to \{\emptyset, X\} \cup f(\widehat{\mathsf{T}})$ such that $\psi^{-1} \circ \psi = \mathrm{id}|\{\emptyset, X\} \cup f(\widehat{\mathsf{T}})$. This function is equivariant, symmetric, and monotone because so is ψ and the family $\widetilde{\mathsf{T}}$ is maximally $\mathcal{I}$-independent.

Then the function $\varphi = \psi^{-1} \circ e_{g(\widetilde{\mathsf{T}})} : \mathsf{P}(X) \to \{\emptyset, X\} \cup f(\widehat{\mathsf{T}})$ is well-defined and belongs to $\mathrm{End}_\lambda^{\mathcal{I}}(\mathsf{P}(X))$ by Corollary 4.4. Since $f = \varphi \circ \psi \circ f = \varphi \circ e_{\widetilde{\mathsf{T}}} \circ g \circ f$, the function f belongs to the minimal ideal of the semigroup $\mathrm{End}_\lambda(\mathsf{P}(X))$.

To prove the "only if" part, take any function $f \in K(\mathrm{End}_\lambda(\mathsf{P}(X)))$ and for the idempotent $e_{\widetilde{\mathsf{T}}} \in \mathrm{End}_\lambda(\mathsf{P}(X))$ find a function $g \in \mathrm{End}_\lambda(\mathsf{P}(X))$ such that $f = g \circ e_{\widetilde{\mathsf{T}}} \circ f$. Now the properties (1), (2) of the function f follow from the corresponding properties of the idempotent $e_{\widetilde{\mathsf{T}}}$. $\qquad\square$

Since the superextension $\lambda(X)$ of a group X is topologically isomorphic to the semigroup $\mathrm{End}_\lambda(\mathsf{P}(X))$, Theorem 13.1 implies the following description of the minimal ideal $K(\lambda(X))$ of $\lambda(X)$.

Corollary 13.2. *For a twinic group X a maximal linked system $\mathcal{L} \in \lambda(X)$ belongs to the minimal ideal $K(\lambda(X))$ of the superextension $\lambda(X)$ if and only if its function representation $\Phi_\mathcal{L}$ satisfies two conditions:*

(1) *the family $\Phi_\mathcal{L}(\widehat{\mathsf{T}})$ is minimal $\widehat{\mathcal{K}}$-covering;*
(2) *$\Phi_\mathcal{L}(\mathsf{P}(X)) \subset \{\emptyset, X\} \cup f(\widetilde{\mathsf{T}})$.*

14. Minimal left ideals of the semigroup $\mathrm{End}_\lambda(\mathsf{P}(X))$

After elaborating the necessary tools in Section 5-13, we now return to the problem of describing the structure of minimal left ideals of the superextension $\lambda(X)$ of a group X. Our strategy is to find a relative small subfamily $\mathsf{F} \subset \mathsf{P}(X)$ such that for the restriction operator $R_\mathsf{F} : \mathrm{End}_\lambda(\mathsf{P}(X)) \to \mathsf{P}(X)^\mathsf{F}$, $R_\mathsf{F} : \varphi \mapsto \varphi|\mathsf{F}$, the compositions $R_\mathsf{F} \circ \Phi : \lambda(X) \to \mathsf{P}(X)^\mathsf{F}$ is injective on some (equivalently, each) minimal left ideal of $\lambda(X)$. For a twinic group X, such a special minimal left ideal will be found in the left ideal $\lambda^\mathcal{I}(X)$ consisting of $\mathcal{I}$-free maximal linked systems on X for a twinic ideal $\mathcal{I}$ on X.

We recall that for a left-invariant ideal $\mathcal{I}$ on a group X the left ideal $\lambda^\mathcal{I}(X)$ is topologically isomorphic to its functional representation $\mathrm{End}_\lambda^\mathcal{I}(\mathsf{P}(X))$, which consists of all $\mathcal{I}$-free functions $\varphi \in \mathrm{End}_\lambda(\mathsf{P}(X))$.

For a subgroup $H \subset X$ a subfamily $\mathsf{F} \subset \mathsf{P}(X)$ is called

- *H-invariant* if $xA \in \mathsf{F}$ for each $x \in H$ and $A \in \mathsf{F}$;
- *symmetric* if $X \setminus A \in \mathsf{F}$ for each $A \in \mathsf{F}$;
- *$\lambda^\mathcal{I}$-invariant* if $\varphi(\mathsf{F}) \subset \mathsf{F}$ for each $\varphi \in \mathrm{End}_\lambda^\mathcal{I}(\mathsf{P}(X))$;
- *$\mathcal{I}$-upper* if F is symmetric and for any subset $A \in \mathsf{F}$ and any twin set $B \in \mathsf{T}$ with $\mathcal{I}\text{-}\mathrm{Fix}^-(A) \subset \mathrm{Fix}^-(B)$ we get $B \in \mathsf{F}$.

Proposition 14.1. *Each $\mathcal{I}$-upper subfamily $\mathsf{F} \subset \mathsf{T}^\mathcal{I}$ is $\lambda^\mathcal{I}$-invariant.*

Proof. Given a function $\varphi \in \mathrm{End}_\lambda^\mathcal{I}(\mathsf{P}(X))$ and a subset $A \in \mathsf{F}$, we need to show that $\varphi(A) \in \mathsf{F}$. This will follow from the definition of an $\mathcal{I}$-upper subfamily as soon as we check that $\varphi(A)$ is a twin set with $\mathcal{I}\text{-}\mathrm{Fix}^-(A) \subset \mathrm{Fix}^-(\varphi(A))$. For any point $x \in \mathcal{I}\text{-}\mathrm{Fix}(A)$, applying to the $\mathcal{I}$-equality $xA =_\mathcal{I} X \setminus A$ the (equivariant symmetric $\mathcal{I}$-free) function φ, we get

$$x\,\varphi(A) = \varphi(xA) = \varphi(X \setminus A) = X \setminus \varphi(A)$$

and thus $x \in \mathrm{Fix}^-(\varphi(A))$. $\qquad\square$

If a family $\mathsf{F} \subset \mathsf{P}(X)$ is $\lambda^\mathcal{I}$-invariant, then the projection

$$\mathrm{End}_\lambda^\mathcal{I}(\mathsf{F}) = \big\{\varphi|\mathsf{F} : \varphi \in \mathrm{End}_\lambda^\mathcal{I}(\mathsf{P}(X))\big\} \subset \mathsf{F}^\mathsf{F} \subset \mathsf{P}(X)^\mathsf{F}$$

of $\mathrm{End}_\lambda^\mathcal{I}(X)$ onto $\mathsf{P}(X)^\mathsf{F}$ is a compact right-topological semigroup and the restriction operator $R_\mathsf{F} : \mathrm{End}_\lambda^\mathcal{I}(\mathsf{P}(X)) \to \mathrm{End}_\lambda^\mathcal{I}(\mathsf{F})$ is a surjective continuous semigroup homomorphism. Then the composition

$$\Phi_\mathsf{F} = R_\mathsf{F} \circ \Phi : \lambda^\mathcal{I}(X) \to \mathrm{End}_\lambda^\mathcal{I}(\mathsf{F})$$

also is a surjective continuous semigroup homomorphism.

In the role of the subfamily F we shall consider the families:

- $\mathsf{pT}^\mathcal{I}$ of $\mathcal{I}$-pretwin subsets of X;
- $\mathsf{T}^\mathcal{I}$ of $\mathcal{I}$-twin subsets of X;
- T of twin subsets of X;
- $\widehat{\mathsf{T}} = \{A \in \mathsf{T} : \mathrm{Fix}^-(A) \in \widehat{\mathcal{K}}\}$ of twin subsets with maximal 2-cogroup;
- $\mathsf{T}_{[K]} = \{A \in \mathsf{T} : \mathrm{Fix}^-(A) \in [K]\}$ where $[K] = \{xKx^{-1} : x \in X\}$ for $K \in \widehat{\mathcal{K}}$;
- $\mathsf{T}_K = \{A \in \mathsf{T} : \mathrm{Fix}^-(A) = K\}$ for $K \in \widehat{\mathcal{K}}$.

The following proposition easily follows from the corresponding definitions.

Proposition 14.2. (1) *The families $\mathsf{pT}^\mathcal{I}$, $\mathsf{T}^\mathcal{I}$, T, and $\widehat{\mathsf{T}}$ are X-invariant and $\mathcal{I}$-upper. Consequently, they are $\lambda^\mathcal{I}$-invariant.*

(2) *For every maximal 2-cogroup $K \in \widehat{\mathcal{K}}$ the families $\mathsf{T}_{[K]}$ and T_K are $\mathcal{I}$-upper. The family $\mathsf{T}_{[K]}$ is X-invariant while T_K is $\mathrm{Stab}(K)$-invariant.*

For any maximal 2-cogroup $K \in \widehat{\mathcal{K}}$ the chain

$$\mathsf{P}(X) \supset \mathsf{pT}^\mathcal{I} \supset \mathsf{T}^\mathcal{I} \supset \mathsf{T} \supset \widehat{\mathsf{T}} \supset \mathsf{T}_{[K]} \supset \mathsf{T}_K$$

of $\lambda^\mathcal{I}$-invariant subsets of $\mathsf{P}(X)$ induces a chain of compact right-topological semigroups

$$\mathrm{End}_\lambda^\mathcal{I}(\mathsf{P}(X)) \to \mathrm{End}_\lambda^\mathcal{I}(\mathsf{pT}^\mathcal{I}) \to \mathrm{End}_\lambda^\mathcal{I}(\mathsf{T}^\mathcal{I}) \to \mathrm{End}_\lambda^\mathcal{I}(\mathsf{T}) \to \mathrm{End}_\lambda^\mathcal{I}(\widehat{\mathsf{T}}) \to \mathrm{End}_\lambda^\mathcal{I}(\mathsf{T}_{[K]}) \to \mathrm{End}_\lambda^\mathcal{I}(\mathsf{T}_K)$$

linked by the restriction operators which are continuous semigroup homomorphisms.

If the ideal $\mathcal{I}$ is twinic, then the families $\mathsf{pT}^{\mathcal{I}}$ and $\mathsf{T}^{\mathcal{I}}$ coincide and so do the semigroups $\mathrm{End}_\lambda^{\mathcal{I}}(\mathsf{pT}^{\mathcal{I}})$ and $\mathrm{End}_\lambda^{\mathcal{I}}(\mathsf{T}^{\mathcal{I}})$.

Now we characterize functions belong to the semigroup $\mathrm{End}_\lambda^{\mathcal{I}}(\mathsf{T}^{\mathcal{I}})$. We shall do that in a more general context of $\mathcal{I}$-upper subfamilies of $\mathsf{T}^{\mathcal{I}}$.

Proposition 14.3. *Let* $\mathsf{F} \subset \mathsf{T}^{\mathcal{I}}$ *be an $\mathcal{I}$-upper subfamily. A function* $\varphi : \mathsf{F} \to \mathsf{F}$ *belongs to the semigroup* $\mathrm{End}_\lambda^{\mathcal{I}}(\mathsf{F})$ *if and only if*

    (1) *φ is equivariant in the sense that $\varphi(xA) = x\varphi(A)$ for all $x \in X$ and $A \in \mathsf{F}$ with $xA \in \mathsf{F}$;*
    (2) *φ is monotone in the sense that $\varphi(A) \subset \varphi(B)$ for any subsets $A, B \in \mathsf{F}$ with $A \subset B$;*
    (3) *φ is $\mathcal{I}$-free in the sense that $\varphi(A) = \varphi(B)$ for any sets $A, B \in \mathsf{F}$ with $A =_{\mathcal{I}} B$;*
    (4) *$\mathcal{I}$-$\mathrm{Fix}^-(A) \subset \mathrm{Fix}^-(\varphi(A))$ for each set $A \in \mathsf{F}$.*

Proof. To prove the "only if" part, take any function $\varphi \in \mathrm{End}_\lambda^{\mathcal{I}}(\mathsf{F})$ and find a function $\bar{\varphi} \in \mathrm{End}_\lambda^{\mathcal{I}}(\mathsf{P}(X))$ such that $\varphi = \bar{\varphi}|\mathsf{F}$.

It follows from Proposition 4.3 that the function φ satisfies the conditions (1)–(3). To check the condition (4), take any set $A \in \mathsf{F} \subset \mathsf{T}^{\mathcal{I}}$ and an element $x \in \mathcal{I}$-$\mathrm{Fix}^-(A)$. Applying the equivariant $\mathcal{I}$-free symmetric function $\bar{\varphi}$ to the $\mathcal{I}$-equality $xA =_{\mathcal{I}} X \setminus A$, we get

$$x\,\bar{\varphi}(A) = \bar{\varphi}(xA) = \bar{\varphi}(X \setminus A) = X \setminus \bar{\varphi}(A)$$

and thus $x \in \mathrm{Fix}^-(\bar{\varphi}(A)) = \mathrm{Fix}^-(\varphi(A))$. This implies the necessary inclusion $\mathcal{I}$-$\mathrm{Fix}^-(A) \subset \mathrm{Fix}^-(\varphi(A))$.

Now we prove the "if" part. Let $\varphi : \mathsf{F} \to \mathsf{F}$ be a function satisfying the conditions (1)–(4). First we prove that the function φ is symmetric. Given arbitrary set $A \in \mathsf{F} \subset \mathsf{T}^{\mathcal{I}}$, find an element $x \in \mathcal{I}$-$\mathrm{Fix}^-(X)$. Applying to the $\mathcal{I}$-equality $xA =_{\mathcal{I}} X\setminus A$ the $\mathcal{I}$-free function φ and taking into account that $x \in \mathcal{I}$-$\mathrm{Fix}^-(A) \subset \mathrm{Fix}^-(\varphi(A))$, we conclude that

$$X \setminus \varphi(A) = x\,\varphi(A) = \varphi(xA) = \varphi(X \setminus A).$$

Now consider the families

$$\mathcal{L}_\varphi = \{x^{-1}A : A \in \mathsf{F},\ x \in \varphi(A)\} \ \text{ and } \ \mathcal{L}_\varphi^{\mathcal{I}} = \{A \subset X : \exists L \in \mathcal{L}_\varphi \ (A =_{\mathcal{I}} L)\}.$$

We claim that the family $\mathcal{L}_\varphi^{\mathcal{I}}$ is linked. Assuming the converse, we could find two sets $A, B \in \mathsf{F}$ and two points $x \in \varphi(A)$ and $y \in \varphi(B)$ such that $x^{-1}A \cap y^{-1}B \in \mathcal{I}$. Then $yx^{-1}A \subset_{\mathcal{I}} X \setminus B$ and hence $yx^{-1}\varphi(A) \subset \varphi(X \setminus B) = X \setminus \varphi(B)$ by the properties (1)–(3) and the symmetry of the function φ. Then $x^{-1}\varphi(A) \subset X \setminus y^{-1}\varphi(B)$, which is not possible because the neutral element e of the group X belongs to $x^{-1}\varphi(A) \cap y^{-1}\varphi(B)$.

Enlarge the $\mathcal{I}$-free linked family $\mathcal{L}_f^{\mathcal{I}}$ to a maximal $\mathcal{I}$-free linked family $\mathcal{L}$, which is maximal linked by Lemma 4.7 and thus $\mathcal{L} \in \lambda^{\mathcal{I}}(X)$. We claim that $\Phi_{\mathcal{L}}|\mathsf{F} = \varphi$. Indeed, take any set $A \in \mathsf{F}$ and observe that

$$\varphi(A) \subset \{x \in X : x^{-1}A \in \mathcal{L}_\varphi\} \subset \{x \in X : x^{-1}A \in \mathcal{L}\} = \Phi_{\mathcal{L}}(A).$$

To prove the reverse inclusion, observe that for any $x \in X \setminus \varphi(A) = \varphi(X \setminus A)$ we get $x^{-1}(X \setminus A) = X \setminus x^{-1}A \in \mathcal{L}_\varphi \subset \mathcal{L}$. Since $\mathcal{L}$ is linked, $x^{-1}A \notin \mathcal{L}$ and hence $x \notin \Phi_{\mathcal{L}}(A)$. $\qquad\square$

For $\mathcal{I}$-independent subfamilies $\mathsf{F} \subset \mathsf{T}^{\mathcal{I}}$ the characterization Proposition 14.3 can be simplified:

Proposition 14.4. *Let* $\mathsf{F} \subset \mathsf{T}^{\mathcal{I}}$ *be an $\mathcal{I}$-independent $\mathcal{I}$-upper subfamily. A function* $\varphi : \mathsf{F} \to \mathsf{F}$ *belongs to the semigroup* $\mathrm{End}_\lambda^{\mathcal{I}}(\mathsf{F})$ *if and only if φ is equivariant, $\mathcal{I}$-free, and $\mathcal{I}$-$\mathrm{Fix}^-(A) \subset \mathrm{Fix}^-(\varphi(A))$ for each set $A \in \mathsf{F}$.*

The following theorem shows that the semigroup $\mathrm{End}_\lambda^{\mathcal{I}}(\widehat{\mathsf{T}})$ has the desired reduction property.

Theorem 14.5. *If a left-invariant ideal $\mathcal{I}$ on a group X is twinic, then the restriction operator $R_{\widehat{\mathsf{T}}} : \mathrm{End}_\lambda^{\mathcal{I}}(\mathsf{P}(X)) \to \mathrm{End}_\lambda^{\mathcal{I}}(\widehat{\mathsf{T}})$ is*

    (1) *injective on each minimal left ideal of the semigroup $\mathrm{End}_\lambda^{\mathcal{I}}(\mathsf{P}(X))$;*
    (2) *bijective on some principal left ideal of $\mathrm{End}_\lambda^{\mathcal{I}}(\mathsf{P}(X))$ under the condition that the family $\widehat{\mathsf{T}}$ is maximally $\mathcal{I}$-independent.*

Proof. 1. Let $\widetilde{\mathsf{T}} \subset \widehat{\mathsf{T}}$ be any minimal $\mathcal{K}$-covering left-invariant subfamily. By Proposition 8.1 and 8.3, the family $\widetilde{\mathsf{T}}$ is maximally $\mathcal{I}$-independent. By Theorem 12.3, there is an idempotent $e_{\widetilde{\mathsf{T}}} \in \mathrm{End}_{\lambda}^{\mathcal{I}}(\mathsf{P}(X))$ such that $e_{\widetilde{\mathsf{T}}}(\mathsf{P}(X) \setminus \mathsf{T}^{\mathcal{I}}) \subset \{\emptyset, X\}$ and $e_{\widetilde{\mathsf{T}}}(\mathsf{T}^{\mathcal{I}}) \subset \widetilde{\mathsf{T}}$. The latter property of $e_{\widetilde{\mathsf{T}}}$ implies that the restriction operator $R_{\widehat{\mathsf{T}}}$ is injective on the principal left ideal $\mathrm{End}_{\lambda}(\mathsf{P}(X)) \circ e_{\widetilde{\mathsf{T}}}$ and consequently, is injective on each minimal left ideal of the semigroup $\mathrm{End}_{\lambda}(\mathsf{P}(X))$ according to Proposition 2.1.

2. To prove the second item, assume that the family $\widehat{\mathsf{T}}$ is maximally $\mathcal{I}$-independent and repeat the above argument for the idempotent $e_{\widehat{\mathsf{T}}}$. $\qquad\square$

Since $\Phi : \lambda(X) \to \mathrm{End}_{\lambda}(\mathsf{P}(X))$ is a topological isomorphism the preceding theorem implies:

Corollary 14.6. *If a left-invariant ideal $\mathcal{I}$ on a group X is twinic, then the function representation $\Phi_{\widehat{\mathsf{T}}} : \lambda^{\mathcal{I}}(X) \to \mathrm{End}_{\lambda}^{\mathcal{I}}(\widehat{\mathsf{T}})$ is*

(1) *injective on each minimal left ideal of the semigroup $\lambda^{\mathcal{I}}(X)$;*

(2) *bijective on some principal left ideal of $\lambda^{\mathcal{I}}(X)$ under the condition that the family $\widehat{\mathsf{T}}$ is maximally $\mathcal{I}$-independent.*

In light of Corollary 14.6, it is important to study the structure of the semigroup $\mathrm{End}_{\lambda}^{\mathcal{I}}(\widehat{\mathsf{T}})$ for a twinic ideal $\mathcal{I}$ on a group X. In this case the semigroup $\mathrm{End}_{\lambda}^{\mathcal{I}}(\widehat{\mathsf{T}})$ has a simple product structure.

Since $\widehat{\mathsf{T}} = \bigcup_{[K] \in \widehat{\mathcal{K}}} \mathsf{T}_{[K]}$, the restriction operators $R_{\mathsf{T}_{[K]}} : \mathrm{End}_{\lambda}^{\mathcal{I}}(\widehat{\mathsf{T}}) \to \mathrm{End}_{\lambda}^{\mathcal{I}}(\mathsf{T}_{[K]})$, $[K] \in [\widehat{\mathcal{K}}]$, compose an injective semigroup homomorphism

$$R_{\mathsf{T}_{[\mathcal{K}]}} : \mathrm{End}_{\lambda}^{\mathcal{I}}(\widehat{\mathsf{T}}) \to \prod_{[K] \in [\widehat{\mathcal{K}}]} \mathrm{End}_{\lambda}^{\mathcal{I}}(\mathsf{T}_{[K]}), \quad R_{\mathsf{T}_{[\mathcal{K}]}} : \varphi \mapsto (\varphi|\mathsf{T}_{[K]})_{[K] \in [\widehat{K}]}.$$

Proposition 14.7. *If the family $\widehat{\mathsf{T}}$ is $\mathcal{I}$-independent, then $R_{\mathsf{T}_{[\mathcal{K}]}} : \mathrm{End}_{\lambda}^{\mathcal{I}}(\widehat{\mathsf{T}}) \to \prod_{[K] \in [\widehat{\mathcal{K}}]} \mathrm{End}_{\lambda}^{\mathcal{I}}(\mathsf{T}_{[K]})$ is a topological isomorphism.*

Proof. We need to show that the map $R_{\mathsf{T}_{[\mathcal{K}]}}$ is surjective. Given any $(f_{[K]})_{[K] \in [\widehat{\mathcal{K}}]} \in \prod_{[K] \in [\widehat{\mathcal{K}}]} \mathrm{End}_{\lambda}^{\mathcal{I}}(\mathsf{T}_{[K]})$, define a function $f : \widehat{\mathsf{T}} \to \widehat{\mathsf{T}}$ letting $f|\mathsf{T}_{[K]} = f_{[K]}$ for $[K] \in [\widehat{\mathcal{K}}]$.

Since the family $\widehat{\mathsf{T}}$ is $\mathcal{I}$-independent, the function f belongs to $\mathrm{End}_{\lambda}^{\mathcal{I}}(\widehat{\mathsf{T}})$ by Proposition 14.4. Since $R_{[\widehat{K}]}(f) = (f_{[K]})_{[K] \in [\widehat{\mathcal{K}}]}$, we see that the homomorphism $R_{[\widehat{\mathcal{K}}]}$ is surjective and hence is a topological isomorphism by the compactness of the semigroup $\mathrm{End}_{\lambda}^{\mathcal{I}}(\widehat{\mathsf{T}})$. $\qquad\square$

Now we see that for understanding the structure of the semigroup $\mathrm{End}_{\lambda}^{\mathcal{I}}(\widehat{\mathsf{T}})$ it is necessary to study the structure of the semigroups $\mathrm{End}_{\lambda}^{\mathcal{I}}(\mathsf{T}_{[K]})$ for $[K] \in [\widehat{\mathcal{K}}]$.

Proposition 14.8. *For any maximal 2-cogroup $K \in \widehat{\mathcal{K}}$ the restriction map*

$$R_{\mathsf{T}_K} : \mathrm{End}_{\lambda}^{\mathcal{I}}(\mathsf{T}_{[K]}) \to \mathrm{End}_{\lambda}^{\mathcal{I}}(\mathsf{T}_K), \quad R_{\mathsf{T}_K} : \varphi \mapsto \varphi|\mathsf{T}_K,$$

is a topological isomorphism.

Proof. Because of the compactness of the semigroup $\mathrm{End}_{\lambda}^{\mathcal{I}}(X, \mathsf{T}_{[K]})$ it suffices to check that the restriction operator $R_{\mathsf{T}_K} : \mathrm{End}_{\lambda}^{\mathcal{I}}(\mathsf{T}_{[K]}) \to \mathrm{End}_{\lambda}^{\mathcal{I}}(\mathsf{T}_K)$ is one-to-one. Given two distinct functions $f, g \in \mathrm{End}_{\lambda}^{\mathcal{I}}(\mathsf{T}_{[K]})$ find a twin set $A \in \mathsf{T}_{[K]}$ such that $f(A) \neq g(A)$. Since $\mathrm{Fix}^-(A) \in [K]$, there is a point $x \in X$ such that $\mathrm{Fix}^-(xA) = x \mathrm{Fix}^-(A) x^{-1} = K$. Consequently, $xA \in \mathsf{T}_K$ and $f(xA) = xf(A) \neq xg(A) = g(xA)$ witnessing that $f|\mathsf{T}_K \neq g|\mathsf{T}_K$. $\qquad\square$

Corollary 14.9. *If the family $\widehat{\mathsf{T}}$ is $\mathcal{I}$-independent, then for any family $\mathcal{K}' \subset \widehat{\mathcal{K}}$ having one-point intersection with each orbit $[K] \in [\widehat{\mathcal{K}}]$ the restriction operator*

$$R_{\mathcal{K}'} : \mathrm{End}_{\lambda}^{\mathcal{I}}(\widehat{\mathsf{T}}) \to \prod_{K \in \mathcal{K}'} \mathrm{End}_{\lambda}^{\mathcal{I}}(\mathsf{T}_K), \quad R_{\mathcal{K}'} : \varphi \mapsto (\varphi|\mathsf{T}_K)_{K \in \mathcal{K}'},$$

is a topological isomorphism of the compact right-topological semigroups.

15. The structure of the semigroups $\mathrm{End}_\lambda^{\mathcal{I}}(\mathsf{T}_K)$

In this section we study the structure of the semigroup $\mathrm{End}_\lambda^{\mathcal{I}}(\mathsf{T}_K)$ where $K \in \widehat{\mathcal{K}}$ is a maximal 2-cogroup and $\mathcal{I}$ is a left-invariant ideal on a group X.

It follows from Proposition 5.4 that for each set $A \in \mathsf{T}_K$ and an element $a \in X$ the shift aA belongs to T_K if and only if $a \in \mathrm{Stab}(K) = \{x \in X : xKx^{-1} = K\}$.

Therefore the subgroup $\mathrm{Stab}(K)$ of X acts on the set T_K by left shifts. The stabilizer $\mathrm{Fix}(A)$ of each set $A \in \mathsf{T}_K$ coincides with $\mathrm{Fix}^-(A) \cdot \mathrm{Fix}^-(A) = K \cdot K$. Since the subgroup KK is normal in $\mathrm{Stab}(K)$, so we can consider the quotient group $H(K) = \mathrm{Stab}(K)/KK$ called the *characteristic group* of K and studied in Section 9. The left action of the group $\mathrm{Stab}(K)$ on T_K induces a free left action of the characteristic group $H(K)$ on T_K: for an element $xKK \in H(K)$ and a twin set $A \in \mathsf{T}_K$ we put $xKK \cdot A = xKKA = xA$ (the latter equality follows from $\mathrm{Fix}(A) = \mathrm{Fix}^-(A) \cdot \mathrm{Fix}^-(A) = K \cdot K$).

So T_K can be (and will be) considered as an $H(K)$-act. Being free, the act T_K is isomorphic to the product $H(K) \times [\mathsf{T}_K]$ endowed with the action $h(x,y) = (hx, y)$ of the group $H(K)$. Let $\mathrm{End}(\mathsf{T}_K)$ denote the endomorphism monoid of the $H(K)$-act. The monoid $\mathrm{End}(\mathsf{T}_K)$ contains a left ideal $\mathrm{End}^{\mathcal{I}}(\mathsf{T}_K)$ consisting of $H(K)$-equivariant maps $\varphi : \mathsf{T}_K \to \mathsf{T}_K$ that are $\mathcal{I}$-free in the sense that $\varphi(A) = \varphi(B)$ for any sets $A =_{\mathcal{I}} B$ in T_K.

Theorem 15.1. *If for a maximal 2-cogroup $K \in \widehat{\mathcal{K}}$ the family T_K is $\mathcal{I}$-independent, then*

(1) $\mathrm{End}_\lambda^{\mathcal{I}}(\mathsf{T}_K) = \mathrm{End}_\lambda(\mathsf{T}_K)$ *if the family $\mathsf{T}^{\mathcal{I}}$ is maximally $\mathcal{I}$-independent;*

(2) *the semigroup $\mathrm{End}_\lambda(\mathsf{T}_K)$ coincides with the endomorphism monoid $\mathrm{End}(\mathsf{T}_K)$ while the semigroup $\mathrm{End}_\lambda^{\mathcal{I}}(\mathsf{T}_K)$ coincides with the left ideal $\mathrm{End}^{\mathcal{I}}(\mathsf{T}_K)$ of endomorphism monoid $\mathrm{End}(\tau_K)$;*

(3) *the semigroup $\mathrm{End}(\mathsf{T}_K)$ is algebraically isomorphic to the wreath product $H(K) \wr [\mathsf{T}_K]^{[\mathsf{T}_K]}$;*

(4) *the minimal ideal $K(\mathrm{End}_\lambda(\mathsf{T}_K))$ coincides with the set $\{f \in \mathrm{End}(\mathsf{T}_K) : \forall A \in f(\mathsf{T}_K), \ f(\mathsf{T}_K) \subset [A]\}$;*

(5) *for each idempotent $f \in \mathrm{End}_\lambda(\mathsf{T}_K)$ the maximal subgroup $H(f) \subset \mathrm{End}_\lambda^{\mathcal{I}}(\mathsf{T}_K)$ containing f is isomorphic to $H(K) \wr S_{[f(\mathsf{T}_K)]}$;*

(6) *each minimal left ideal of the semigroup $\mathrm{End}_\lambda^{\mathcal{I}}(\mathsf{T}_K)$ is isomorphic to $H(K) \times [\mathsf{T}_K]$ where the orbit space $[\mathsf{T}_K]$ is endowed with the left zero multiplication;*

(7) *each maximal subgroup of the minimal ideal of $\mathrm{End}_\lambda^{\mathcal{I}}(\mathsf{T}_K)$ is algebraically isomorphic to $H(K)$;*

(8) *for each minimal idempotent $f \in \mathrm{End}_\lambda^I(\mathsf{T}_K)$ the maximal subgroup $H(f) = f \circ \mathrm{End}_\lambda^{\mathcal{I}}(\mathsf{T}_K) \circ f$ is topologically isomorphic to the twin-generated group $H(A)$ where $A \in f(\mathsf{T}_K)$.*

Proof. 1. Assume that the family T_K is maximally $\mathcal{I}$-independent. We need to show that each function $\varphi \in \mathrm{End}_\lambda^{\mathcal{I}}(\mathsf{T}_K)$ is $\mathcal{I}$-free. Take any sets $A =_{\mathcal{I}} B$ in T_K. Since $\mathcal{I}$ is maximally independent, $A = B$ and thus $\varphi(A) = \varphi(B)$.

2. The equalities $\mathrm{End}_\lambda(\mathsf{T}_K) = \mathrm{End}(\mathsf{T}_K)$ and $\mathrm{End}_\lambda^{\mathcal{I}}(\mathsf{T}_K) = \mathrm{End}^{\mathcal{I}}(\mathsf{T}_K)$ follow from Proposition 14.4.

3–7. The statements (3)–(7) follow from Theorem 3.1 and the fact that $\mathrm{End}^{\mathcal{I}}(\mathsf{T}_K)$ is a left ideal in $\mathrm{End}(\mathsf{T})$.

8. Given a minimal idempotent $f \in \mathrm{End}_\lambda^{\mathcal{I}}(\mathsf{T}_K)$ we shall show that the maximal subgroup $H(f) = f \circ \mathrm{End}_\lambda(\mathsf{T}_K) \circ f$ is topologically isomorphic to the characteristic group $H(A)$ of any twin set $A \in f(\mathsf{T}_K)$.

We recall that $H(A)$ is the characteristic group $H(K)$ of the 2-cogroup $K = \mathrm{Fix}^-(A)$, endowed with the topology generated by the twin set $q(A \cap \mathrm{Stab}(K))$ where $q : \mathrm{Stab}(K) \to H(K) = \mathrm{Stab}(K)/KK$ is the quotient homomorphism.

We define a topological isomorphism $\Theta_A : H(f) \to H(A)$ from the maximal subgroup $H(f) = f \cdot \mathrm{End}_\lambda(\mathsf{T}_K) \cdot f$ of f to the twin-generated group $H(A)$ in the following way. By Theorem 13.1(1) and Proposition 7.6, for any function $g \in H(f)$, the set $g(A) = fgf(A) \in [A]$, so we can find $x \in X$ with $fgf(A) = x^{-1}A$. Since $\mathrm{Fix}^-(A) \subset \mathrm{Fix}^-(fgf(A)) = \mathrm{Fix}^-(x^{-1}A)$, by the maximality of $K = \mathrm{Fix}^-(A) \in \widehat{\mathcal{K}}$, we conclude that $x^{-1}\mathrm{Fix}^-(A)x = \mathrm{Fix}^-(x^{-1}A) = \mathrm{Fix}^-(g(A)) = \mathrm{Fix}^-(A)$ and thus $x \in \mathrm{Stab}(K)$. So, it is legal to define $\Theta_A(g)$ as the image $q(x) = xKK = KKx$ of x under the quotient homomorphism $q : \mathrm{Stab}(K) \to H(K) = H(A)$.

It remains to prove that $\Theta_A : H(f) \to H(A)$ is a well-defined topological isomorphism of the right-topological groups.

First we check that Θ_A is well-defined, that is $\Theta_A(g) = q(x^{-1})$ does not depend on the choice of the point x. Indeed, for any other point $y \in X$ with $g(A) = y^{-1}A$ we get $x^{-1}A = y^{-1}A$ and thus $yx^{-1} \in \mathrm{Fix}(A) = K \cdot K$ where $K = \mathrm{Fix}^-(A)$. Consequently, $q(x) = KKx = KKy = q(y)$.

Next, we prove that Θ_A is a group homomorphism. Given two functions $g, h \in H(f)$, find elements $x_g, x_h \in \mathrm{Fix}(X)$ such that $h(A) = x_h^{-1}A$ and $g(A) = x_g^{-1}A$. It follows that $g \circ h(A) = g(x_h^{-1}A) = x_h^{-1}g(A) = x_h^{-1}x_g^{-1}(A) = (x_gx_h)^{-1}A$, which implies that $\Theta_A(g \circ h) = x_gx_hKK = \Theta_A(g) \cdot \Theta_A(h)$.

Now, we calculate the kernel of the homomorphism Θ_A. Take any function $g \in H(f)$ with $\Theta_A(g) = e$, which means that $g(A) = fgf(A) = A$. Then for every $A' \in \mathsf{T}_K$ we can find $x \in X$ with $f(A') = xA$ and conclude that $g(A') = fgf(A') = fg(xA) = xfg(A) = xfgf(A) = xA = f(A')$ witnessing that $g = fgf = f$. This means that the homomorphism Θ_A is one-to-one.

To see that Θ_A is onto, first observe that each element of the characteristic group $H(A)$ can be written as $[y] = yKK = KKy \in H(K)$ for some $y \in \mathrm{Stab}(K)$. Given such an element $[y] \in H(A)$, consider the equivariant function $s_{[y]} : [A] \to [A]$, $s_{[y]} : zA \mapsto zy^{-1}A = zy^{-1}KKA$. Let us show that this function is well defined. Indeed, for each point $u \in X$ with $zA = uA$, we get $u^{-1}z \in \mathrm{Fix}(A)$ and hence, $yu^{-1}zy^{-1} \in y\mathrm{Fix}(A)y^{-1} = yKKy^{-1} = KK = \mathrm{Fix}(A)$. Then $yu^{-1}zy^{-1}A = A$ and hence $zy^{-1}A = uy^{-1}A$.

By Proposition 14.4, the composition $s_{[y]} \circ f$, being equivariant and $\mathcal{I}$-free, belongs to $\mathrm{End}_\lambda^{\mathcal{I}}(\mathsf{T}_K)$. It follows from $s_{[y]} \circ f = f \circ s_{[y]} \circ f$ that the function $s_{[y]} \circ f$ belongs to the maximal group $H(f)$. Since $s_{[y]} \circ f(A) = s_{[y]}(A) = y^{-1}A$, the image $\Theta_A(s_{[y]} \circ f) = [y]$. So, $\Theta_A(H(f)) = H(A)$ and $\Theta_A : H(f) \to H(A)$ is an algebraic isomorphism.

It remains to prove that this isomorphism is topological. Observe that for every $[y] \in H(A)$ we get $s_{[y]} \circ f(A) = s_{[y]}(A) = y^{-1}KKA = y^{-1}A$. Consequently, $x \in s_{[y]} \circ f(A)$ iff $x \in y^{-1}A$ iff $y \in Ax^{-1}$.

To see that the map $\Theta_A : H(f) \to H(A)$ is continuous, take any sub-basic open set

$$U_x = \{[y] \in H(A) : y \in Ax^{-1}\}, \quad x \in \mathrm{Stab}(K),$$

in $H(A)$ and observe that $\Theta_A^{-1}(U_x) = \{s_{[y]} \circ f : [y] \in U_x\} = \{s_{[y]} \circ f : y \in Ax^{-1}\} = \{s_{[y]} \circ f : x \in s_{[y]} \circ f(A)\}$ is a sub-basic open set in $H(f)$. To see that the inverse map $\Theta_A^{-1} : H(A) \to H(f)$ is continuous, take any sub-basic open set $V_{x,T} = \{g \in H(f) : x \in g(T)\}$ where $x \in X$ and $T \in \mathsf{T}_K$. It follows that $f(T) = x_TA$ for some $x_T \in X$. Then

$$\Theta_A(V_{x,T}) = \{[y] \in H(A) : x \in s_{[y]} \circ f(T)\} = \{[y] \in H(A) : x \in s_{[y]}(x_TA)\} =$$
$$= \{[y] \in H(A) : x_T^{-1}x \in s_{[y]}(A)\} = \{[y] \in H(A) : y \in Ax^{-1}x_T\}$$

is a sub-basic open set in $H(A)$. $\qquad \square$

In the following proposition we calculate the cardinalities of the objects appearing in Theorem 15.1.

Proposition 15.2. *If the index $m = |X/K|$ of a maximal 2-cogroup $K \in \widehat{\mathcal{K}}$ in X is finite, then*

(1) *the family T_K is maximally $\mathcal{I}$-independent;*
(2) $|\mathsf{T}_K| = 2^{m/2}$;
(3) $|H(K)| = 2^k$ *for some k such that 2^k divides m;*
(4) $|[\mathsf{T}_K]| = 2^{m/2-k}$;
(5) $\mathrm{End}_\lambda^{\mathcal{I}}(\mathsf{T}_K) = \mathrm{End}(\mathsf{T}_K)$ *is a finite semigroup of cardinality*
$|\mathrm{End}_\lambda^{\mathcal{I}}(\mathsf{T}_K)| = |H(K) \wr [\mathsf{T}_K]^{[\mathsf{T}_K]}| = |H(K)^{[\mathsf{T}_K]} \times [\mathsf{T}_K]^{[\mathsf{T}_K]}| = 2^{m2^{m/2-k-1}}$.

Proof. Assume that the index $m = |X/K|$ of a maximal 2-cogroup $K \in \widehat{\mathcal{K}}$ is finite. Consider the group $K^\pm = K \cup KK$ and take a subset $S \subset X$ that meets each coset $K^\pm x$, $x \in X$, at a single point. It follows that $|S| = m/2$.

1. By Proposition 8.2, the family T_K is maximally $\mathcal{I}$-independent and then $\mathrm{End}_\lambda^{\mathcal{I}}(\mathsf{T}_K) = \mathrm{End}(\mathsf{T}_K)$ according to Theorem 15.1(1).

2. For every subset $E \subset S$, consider the twin set $T_E = KKE \cup K \cdot (S \setminus E)$. It is easy to check that $\mathsf{T}_K = \{T_E : E \subset S\}$ and hence $|\mathsf{T}_K| = 2^{|S|} = 2^{m/2}$.

3. Observe that the orbit $\{xT_E : x \in \mathrm{Stab}(K)\}$ of each set T_E in T_K consists of $|H(K)| = |\mathrm{Stab}(K)/KK|$ elements. Consequently, $|H(K)|$ divides $2^{m/2} = |\mathsf{T}_K|$ and hence $|H(K)| = 2^k$ for some $k \leq m/2$. Since $\mathrm{Stab}(K)$ is a subgroup of X, $2^k = |H(K)| = |\mathrm{Stab}(K)/KK|$ divides $|X/KK| = |X/K| = m$.

4. It follows that $|[\mathsf{T}_K]| = |\mathsf{T}_K|/|H(K)| = 2^{m/2-k}$.

5. By Theorem 15.1(3), the semigroup $\mathrm{End}_\lambda(\mathsf{T}_K)$ is isomorphic to $H(K) \wr [\mathsf{T}_K]^{[\mathsf{T}_K]}$ and hence has cardinality

$$|\mathrm{End}_\lambda(\mathsf{T}_K)| = (2^k)^{2^{m/2-k}} \cdot (2^{m/2-k})^{2^{m/2-k}} = 2^{m2^{m/2-k-1}}.$$

$\square$

16. Continuity of the semigroup operation of $\mathrm{End}_\lambda(\mathsf{F})$

In this section we study the problem of the continuity of the semigroup operation on $\mathrm{End}_\lambda(\mathsf{T}_K) = \mathrm{End}_\lambda(\mathsf{T}_{[K]})$ for $K \in \widehat{\mathcal{K}}$. This will be done in a more general context of upper subfamilies $\mathsf{F} \subset \mathsf{T}$. We recall that a family $\mathsf{F} \subset \mathsf{T}$ is *upper* if F is symmetric and for any twin set $A \in \mathsf{F}$ and a twin subset $B \subset X$ with $\mathrm{Fix}^-(A) \subset \mathrm{Fix}^-(B)$, we get $B \in \mathsf{F}$.

Let us remark that $\widetilde{\mathsf{T}}$ is an upper subfamily of T while T_K is a minimal upper subfamily of T for every $K \in \widetilde{\mathcal{K}}$.

We recall that a right-topological semigroup S is called *semi-topological* if the semigroup operation $S \times S \to S$ is separately continuous. If the semigroup operation is continuous, then S is called a *topological semigroup*.

Theorem 16.1. *For an left-invariant upper subfamily $\mathsf{F} \subset \mathsf{T}$ the following conditions are equivalent:*

  (1) $\mathrm{End}_\lambda(\mathsf{F})$ *is a topological semigroup;*
  (2) $\mathrm{End}_\lambda(\mathsf{F})$ *is a semi-topological semigroup;*
  (3) *for each twin set $A \in \mathsf{F}$ the subgroup $\mathrm{Fix}(A)$ has finite index in X.*

Proof. (3) $\Rightarrow$ (1) Assume that for each twin set $A \in \mathsf{F}$ the stabilizer $\mathrm{Fix}(A)$ has finite index in X. To show that the semigroup operation $\circ : \mathrm{End}_\lambda(\mathsf{F}) \times \mathrm{End}_\lambda(\mathsf{F}) \to \mathrm{End}_\lambda(\mathsf{F})$ is continuous, fix any two functions $f, g \in \mathrm{End}_\lambda(\mathsf{F})$ and a neighborhood $O(f \circ g)$ of their composition. We should show that the functions f, g have neighborhoods $O(f), O(g) \subset \mathrm{End}_\lambda(\mathsf{F})$ such that $O(f) \circ O(g) \subset O(f \circ g)$. We lose no generality assuming that the neighborhood $O(f, g)$ is of sub-basic form:

$$O(f \circ g) = \{h \in \mathrm{End}_\lambda(\mathsf{F}) : x \in h(A)\}$$

for some $x \in X$ and some twin set $A \in \mathsf{F}$. Let $B = g(A)$. It follows from $f \circ g \in O(f \circ g)$ that $x \in f \circ g(A) = f(B)$. Let $O(f) = \{h \in \mathrm{End}_\lambda(\mathsf{F}) : x \in h(B)\}$.

The definition of a neighborhood $O(g)$ is a bit more complicated. By our hypothesis, the stabilizer $\mathrm{Fix}(A)$ has finite index in X. Let $S \subset X$ be a (finite) subset meeting each coset $z\mathrm{Fix}(A)$, $z \in X$, at a single point. Consider the following open neighborhood of g in $\mathrm{End}_\lambda(\mathsf{F})$:

$$O(g) = \{g' \in \mathrm{End}_\lambda(\mathsf{F}) : \forall s \in S \; (s \in B \; \Leftrightarrow \; s \in g'(A))\}.$$

We claim that $O(f) \circ O(g) \subset O(f \circ g)$. Indeed, take any functions $f' \in O(f)$ and $g' \in O(g)$. By Proposition 14.3(4), $\mathrm{Fix}^-(A) \subset \mathrm{Fix}^-(g'(A))$ and hence $\mathrm{Fix}(A) \subset \mathrm{Fix}(g'(A))$. Then $g'(A) = (S \cap g'(A)) \cdot \mathrm{Fix}(A) = (S \cap B) \cdot \mathrm{Fix}(A) = B$ and then $x \in f'(B) = f' \circ g'(A)$ witnessing that $f' \circ g' \in O(f \circ g)$.

The implication (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (3) Assume that X contains a twin subset $T_0 \in \mathsf{F}$ whose stabilizer $\mathrm{Fix}(T_0)$ has infinite index in X. Then the subgroup $H = \mathrm{Fix}^\pm(T_0)$ also has infinite index in X. Fix any point $c \in \mathrm{Fix}^-(T_0)$.

Since H has infinite index in X, we can apply Theorem 15.5 of [16] and conclude that $X \neq FHF$ for any finite subset $F \subset X$.

Lemma 16.2. *There are countable sets $A, B \subset X$ such that*

  (1) $xB \cap yB = \emptyset$ *for any distinct $x, y \in A$;*
  (2) $|AB \cap Hz| \leq 1$ *for all $z \in X$;*
  (3) $e \in A$, $AB \cap H = \emptyset$.

Proof. Let $a_0 = e$ and $B_{<0} = \{e\}$. Inductively we shall construct sequences $A = \{a_n : n \in \omega\}$ and $B = \{b_n : n \in \omega\}$ such that

- $b_n \notin A_{\leq n}^{-1} H A_{\leq n} B_{<n}$ where $A_{\leq n} = \{a_i : i \leq n\}$ and $B_{<n} = \{b_i : i < n\}$;
- $a_{n+1} \notin H A_{\leq n} B_{\leq n} B_{\leq n}^{-1}$.

Since $X \neq FHF$ for any finite subset $F \subset X$, the choice of the points b_n and a_{n+1} at the n-th step is always possible. It is easy to check that the sets A, B satisfy the conditions (1)–(3) of the lemma. $\square$

The properties (2), (3) of the set AB allows us to enlarge AB to a subset S that contains the neutral element of X and meets each coset Hz, $z \in X$, at a single point. Observe that each subset $E \subset S$ generates a twin subset

$$T_E = \mathrm{Fix}(T_0) \cdot E \cup \mathrm{Fix}^-(T_0) \cdot (S \setminus E)$$

of X such that $\mathrm{Fix}^-(T_0) \subset \mathrm{Fix}^-(T_E)$ and hence $T_E \in \mathsf{F}$.

Lemma 16.3. *There is a free ultrafilter $\mathcal{B}$ on X and a family of subsets $\{U_a : a \in A\} \subset \mathcal{B}$ such that*

(1) $\bigcup_{a \in A} U_a \subset B$;
(2) *the set $U = \bigcup_{a \in A} aU_a$ has the property $B \not\subset x^{-1}U \cup y^{-1}U$ for every $x, y \in A$;*
(3) *for every $V \in \mathcal{B}$ the set $\{a \in A : xV \subset U\}$ is finite.*

Proof. Let $A = \{a_n : n \in \omega\}$ and B be the sets constructed in Lemma 16.2. For every $n \in \omega$ let $A_{\leq n} = \{a_i : i \leq n\}$. Let $B_{<0} = \{e\}$ and inductively, for every $n \in \omega$ choose an element $b_n \in B$ so that

$$b_n \notin A_{\leq n}^{-1} A_{\leq n} B_{<n} \quad \text{where} \quad B_{<n} = \{b_i : i < n\}.$$

For every $n \in \omega$ let $B_{\geq n} = \{b_i : i \geq n\}$. Let also $B_{2\omega} = \{b_{2n} : n \in \omega\}$.

Let us show that for any distinct numbers n, m the intersection $a_n B_{\geq n} \cap a_m B_{\geq m}$ is empty. Otherwise there would exist two numbers $i \geq n$ and $j \geq m$ such that $a_n b_i = a_m b_j$. It follows from $a_n \neq a_m$ that $i \neq j$. We lose no generality assuming that $j > i$. Then $a_n b_i = a_m b_j$ implies that

$$b_j = a_m^{-1} a_n b_i \in A_{\leq j}^{-1} A_{\leq j} B_{<j},$$

which contradicts the choice of b_j.

Let $\mathcal{B} \in \beta(X)$ be any free ultrafilter such that $B_{2\omega} \in \mathcal{B}$ and $\mathcal{B}$ is not a P-point. To get such an ultrafilter, take $\mathcal{B}$ to be a cluster point of any countable subset of $\beta(B_{2\omega}) \setminus B_{2\omega} \subset \beta(X)$. Using the fact that $\mathcal{B}$ fails to be a P-point, we can take a decreasing sequence of subsets $\{V_n : n \in \omega\} \subset \mathcal{B}$ of $B_{2\omega}$ having no pseudointersection in $\mathcal{B}$. The latter means that for every $V \in \mathcal{B}$ the almost inclusion $V \subset^* V_n$ (which means that $V \setminus V_n$ is finite) holds only for finitely many numbers n.

For every $a = a_n \in A$ let $U_a = V_n \cap B_{\geq n}$. We claim that the ultrafilter $\mathcal{B}$, the family $(U_a)_{a \in A}$, and the set $U = \bigcup_{a \in A} aU_a = \bigcup_{n \in \omega} a_n(V_n \cap B_{\geq n})$ satisfy the requirements of the lemma.

First, we check that $B \not\subset a_n^{-1}U \cup a_m^{-1}U$ for all $n \leq m$. Take any odd number $k > m$. We claim that $b_k \notin a_n^{-1}U \cup a_m^{-1}U$. Otherwise, $b_k \in a_n^{-1} a_i(V_i \cap B_{\geq i}) \cup a_m^{-1} a_i(V_i \cap B_{\geq i})$ for some $i \in \omega$ and hence $b_k = a_n^{-1} a_i b_j$ or $b_k = a_m^{-1} a_i b_j$ for some even $j \geq i$. If $k > j$, then both the equalities are forbidden by the choice of $b_k \notin A_{\leq k}^{-1} A_{\leq k} B_{<k} \supset \{a_n^{-1} a_i b_j, a_m^{-1} a_i b_j\}$. If $k < j$, then those equalities are forbidden by the choice of $b_j \notin A_{\leq j}^{-1} A_{\leq j} B_{<j} \supset \{a_i^{-1} a_n b_k, a_i^{-1} a_m b_k\}$. Therefore, $B \not\subset a_n^{-1}U \cup a_m^{-1}U$.

Next, given arbitrary $V \in \mathcal{B}$ we show that the set $A' = \{a \in A : aV \subset U\}$ is finite. By the choice of the sequence (V_n), the set $F = \{a_n : V \cap B_{2\omega} \subset^* V_n\}$ is finite. We claim that $A' \subset F$. Indeed, take any $a_n \in A'$. It follows from $a_n V \subset U = \bigcup_{a \in A} aB_a$ and $a_n B \cap \bigcap_{i \neq n} a_i B = \emptyset$ that

$$a_n(V \cap B_{2\omega}) \subset^* a_n(V_n \cap B_{\geq n}) \subset a_n V_n$$

and hence $a_n \in F$. $\square$

Let $\mathcal{A}$ be any free ultrafilter on X containing the set A, and $\alpha = \Phi_\mathsf{F}(\mathcal{A})$, $\beta = \Phi_\mathsf{F}(\mathcal{B})$ be the function representations of the ultrafilters $\mathcal{A}$ and $\mathcal{B}$. We claim that the left shift $l_\alpha : \mathrm{End}_\lambda(\mathsf{F}) \to \mathrm{End}_\lambda(\mathsf{F})$, $l_\alpha : f \mapsto \alpha \circ f$, is discontinuous at β. Since $U \subset AB \subset S$, we can consider the twin set

$$T = \mathrm{Fix}(T_0) \cdot U \cup \mathrm{Fix}^-(T_0) \cdot (S \setminus U)$$

and observe that $T \in \mathcal{A} \circ \mathcal{B}$. Consequently, $\alpha \circ \beta(T) = \{x \in G : x^{-1}T \in \mathcal{A} \circ \mathcal{B}\}$ contains the neutral element, which implies that $O(\alpha \circ \beta) = \{f \in \mathrm{End}_\lambda(\mathsf{F}) : e \in f(T)\}$ is a neighborhood of $l_\alpha(\beta) = \alpha \circ \beta$ in $\mathrm{End}_\lambda(\mathsf{F})$.

Assuming that l_α is continuous at β, we can find a neighborhood $O(\beta) \subset \mathrm{End}_\lambda(\mathsf{F})$ of β such that $l_\alpha(O(\beta)) \subset O(\alpha \circ \beta)$. Since F is left-invariant, we can assume that $O(\beta)$ is of the basic form:

$$O(\beta) = \{f \in \mathrm{End}_\lambda(\mathsf{F}) : e \in \bigcap_{i=1}^{n} f(T_i)\}$$

for some twin sets $T_1, \ldots, T_n \in \mathsf{F}$. It follows from $\beta \in O(\beta)$ that $e \in \beta(T_i)$ and thus $T_i \in \mathcal{B}$ for every $i \leq n$. According to Lemma 16.3(3), the set $F = \{a \in A : B \cap \bigcap_{i=1}^{n} T_i \subset a^{-1}U\}$ is finite.

We claim that the family $\mathcal{L} = \{T_1, \ldots, T_n, X \setminus x^{-1}T : x \in A \setminus F\}$ is linked. This will follow as soon as we check that

   (i) $T_i \cap (X \setminus x^{-1}T) \neq \emptyset$ for any $i \leq n$ and $x \in A \setminus F$;
   (ii) $(X \setminus x^{-1}T) \cap (X \setminus y^{-1}T) \neq \emptyset$ for all $x, y \in A$.

The item (i) is equivalent to $T_i \not\subset x^{-1}T$ for $x \in A \setminus F$. Assuming conversely that $T_i \subset x^{-1}T$, we will consecutively get $xT_i \subset T$, $S \cap xT_i \subset S \cap T = U$, and finally $B \cap T_i \subset x^{-1}S \cap T_i \subset x^{-1}U$, which contradicts $x \notin F$.

The item (ii) is equivalent to $x^{-1}T \cup y^{-1}T \neq X$ for $x, y \in A$. Assume conversely that $x^{-1}T \cup y^{-1}T = X$ for some $x, y \in A$. It follows from $xB \subset S$ that $xB \cap T = xB \cap U$ and thus $B \cap x^{-1}T = B \cap x^{-1}U$. Similarly, $B \cap y^{-1}T = B \cap y^{-1}U$. Consequently,

$$B = B \cap X = B \cap (x^{-1}T \cup y^{-1}T) = B \cap (x^{-1}U \cup y^{-1}U) \neq B$$

according to Lemma 16.3(2). This contradiction completes the proof of the linkedness of $\mathcal{L}$.

Being linked, the family $\mathcal{L}$ can be enlarged to a maximal linked system $\mathcal{C} \in \lambda(X)$. It follows from $T_1, \ldots, T_n \in \mathcal{L} \subset \mathcal{C}$ that the twin representation $\gamma = \Phi_{\mathsf{F}}(\mathcal{C})$ belongs to the neighborhood $O(\beta)$ and consequently, $\alpha \circ \gamma \in O(\alpha \circ \beta)$, which means that $T \in \mathcal{A} \circ \mathcal{C}$. The latter is equivalent to $A' = \{x \in X : x^{-1}T \in \mathcal{C}\} \in \mathcal{A}$. On the other hand, $X \setminus A' = \{x \in X : X \setminus x^{-1}T \in \mathcal{C}\}$ contains the set $A \setminus F \in \mathcal{A}$ and thus $X \setminus A' \in \mathcal{A}$, which is a contradiction. $\qquad\square$

Theorem 16.4. *For an upper subfamily $\mathsf{F} \subset \mathsf{T}$ the following conditions are equivalent:*

   (1) $\mathrm{End}_\lambda(\mathsf{F})$ *is metrizable;*
   (2) $\mathrm{End}_\lambda(\mathsf{F})$ *is a metrizable topological semigroup;*
   (3) F *is at most countable.*

Proof. We shall prove the implications $(3) \Rightarrow (2) \Rightarrow (1) \Rightarrow (3)$.

$(3) \Rightarrow (2)$. Assume that the family F is at most countable. We claim that for each twin subset $T \in \mathsf{F}$ the 2-cogroup $K = \mathrm{Fix}^-(T)$ has finite index in X. Otherwise, for the subgroup $K^\pm = KK \cup K$ we could find an infinite subset $S \subset X$ meeting each coset $K^\pm x$, $x \in X$, at a single point. Then for every subset $E \subset S$ the set

$$T_E = \mathrm{Fix}(T) \cdot E \cup \mathrm{Fix}^-(T) \cdot (S \setminus E)$$

is a twin set with $\mathrm{Fix}^-(T) \subset \mathrm{Fix}^-(T_E)$. Since F is an upper family containing the twin set T, we get that $T_E \in \mathsf{F}$ for every $E \subset S$. Since the number of subsets E of S is uncountable, we arrive to an absurd conclusion that $|\mathsf{F}| \geq |\{T_E : E \subset S\}|$ is uncountable.

This contradiction shows that $\mathrm{Fix}(T)$ has finite index in X. In this case the implication $(3) \Rightarrow (1)$ of Theorem 16.1 guarantees that $\mathrm{End}_\lambda(\mathsf{F})$ is a topological semigroup. Now we show that this semigroup is metrizable. First observe that for every $T \in \mathsf{F}$ the set

$$\mathrm{End}_\lambda(T, \mathsf{P}(X)) = \{\varphi(T) : \varphi \in \mathrm{End}_\lambda(\mathsf{P}(X))\} \subset \{A \in \mathsf{T} : \mathrm{Fix}(A) \supset \mathrm{Fix}(T)\}$$

is finite. Since the family F is countable, the space $\mathrm{End}_\lambda(\mathsf{F}) \subset \prod_{T \in \mathsf{F}} \mathrm{End}_\lambda(T, \mathsf{P}(X))$ is metrizable, being a subspace of the countable product of finite discrete spaces.

The implication $(2) \Rightarrow (1)$ is trivial.

$(1) \Rightarrow (3)$ Assuming that the family F is not countable, we shall show that the space $\mathrm{End}_\lambda(\mathsf{F})$ is not metrizable. We consider two cases.

(a) For each $T \in \mathsf{F}$ the subgroup $\mathrm{Fix}(T)$ has finite index in X. This implies that the orbit $[T] = \{xT : x \in X\}$ is finite. Let F' be a subset of F meeting each orbit $[T]$, $T \in \mathsf{F}$, at a single point. Since all those orbits are finite and F is uncountable, the set F' is uncountable. By Propositions 8.2, the family F is maximally $\{\emptyset\}$-independent and by Proposition 14.4, the space $\mathrm{End}_\lambda(\mathsf{F})$ is homeomorphic to the uncountable product $\prod_{T \in \mathsf{F}'} \mathrm{End}_\lambda(T, \mathsf{P}(X))$ where for each $T \in \mathsf{F}'$ the space

$$\mathrm{End}_\lambda(T, \mathsf{P}(X)) = \{\varphi(T) : \varphi \in \mathrm{End}_\lambda(\mathsf{F})\} = \{A \in \mathsf{T} : \mathrm{Fix}^-(A) \supset \mathrm{Fix}^-(T)\}$$

contains at least two distinct subsets: T and $X \setminus T$. Now we see that $\mathrm{End}_\lambda(\mathsf{F})$ is not metrizable, being homeomorphic to the uncountable product of non-degenerate spaces.

(b) For some twin set $T \in \mathsf{F}$ the stabilizer $\mathrm{Fix}(T)$ has infinite index. The we can find an infinite set $S \subset X$ that intersects each coset $\mathrm{Fix}^\pm(T)x$, $x \in X$, at a single point. As we already know, for each subset $E \subset S$ the set

$$T_E = \mathrm{Fix}(T) \cdot E \cup \mathrm{Fix}^-(T) \cdot (S \setminus E)$$

belongs to the family F. Now take any two distinct ultrafilters $\mathcal{U}, \mathcal{V} \in \beta(S) \subset \beta(X)$ and consider their function representations $f_\mathcal{U} = \Phi_\mathsf{F}(\mathcal{U})$ and $f_\mathcal{V} = \Phi_\mathsf{F}(\mathcal{V})$. Since $\mathcal{U} \neq \mathcal{V}$, there is a subset $E \subset S$ such that $E \in \mathcal{U} \setminus \mathcal{V}$. It follows that $T_E \in \mathcal{U}$ and $T_{S \setminus E} \in \mathcal{V}$, which implies $e \in f_\mathcal{U}(T_E) \setminus f_\mathcal{V}(T_E)$. This means that $f_\mathcal{U} \neq f_\mathcal{V}$ and consequently, $|\mathrm{End}_\lambda(\mathsf{F})| \geq |\beta(S)| \geq 2^\mathfrak{c}$, which implies that the compact space $\mathrm{End}_\lambda(\mathsf{F})$ is not metrizable (because metrizable compact have cardinality $\leq \mathfrak{c}$). $\qquad\square$

The following proposition characterizes groups containing only countably many twin subsets. Following [4], we define a group X to be *odd* if each element $x \in X$ has odd order.

Proposition 16.5. *The family* T *of twin subsets of a group X is at most countable if and only if each subgroup of infinite index in X is odd.*

Proof. Assume that each subgroup of infinite index in X is odd. We claim that for every $A \in \mathsf{T}$ the subgroup $\mathrm{Fix}(A)$ has finite index in X. Take any point $c \in \mathrm{Fix}^-(A)$ and consider the cyclic subgroup $c^\mathbb{Z} = \{c^n : n \in \mathbb{Z}\}$ generated by c. The subgroup $c^\mathbb{Z}$ has finite index in X, being non-odd. Since $c^{2\mathbb{Z}} = \{c^{2n} : n \in \mathbb{Z}\} \subset \mathrm{Fix}(A)$, we conclude that $\mathrm{Fix}(A)$ also has finite index in X.

Next, we show that the family $\{\mathrm{Fix}(A) : A \in \mathsf{T}\}$ is at most countable. This is trivially true if $\mathsf{T} = \emptyset$. If $\mathsf{T} \neq \emptyset$, then we can take any $A \in \mathsf{T}$ and choose a point $c \in \mathrm{Fix}^-(A)$. The cyclic subgroup $c^\mathbb{Z}$ generated by c is not odd and hence has finite index in X. Consequently, the group X is at most countable. Now it remains to check that for every $x \in X$ the set $\mathsf{T}_x = \{A \in \mathsf{T} : x \in \mathrm{Fix}^-(A)\}$ is finite. If the set T_x is not empty, then the cyclic subgroup $x^\mathbb{Z}$ generated by x is not odd and hence has finite index in G. Consider the subgroup $x^{2\mathbb{Z}}$ of index 2 in $x^\mathbb{Z}$. It is clear that $x^{2\mathbb{Z}} \subset \mathrm{Fix}(A)$. Let $S \subset X$ be a finite set containing the neutral element of X and meeting each coset $x^{2\mathbb{Z}}z$, $z \in X$ at a single point. It follows from $x^{2\mathbb{Z}} \subset \mathrm{Fix}(A)$ that $A = x^{2\mathbb{Z}} \cdot (S \cap A)$ and consequently $|\mathsf{T}_x| \leq 2^{|S|} < \infty$.

Now assume that some subgroup H of infinite index in X is not odd. Then H contains an element $c \in H$ such that the sets $c^{2\mathbb{Z}} = \{c^{2n} : n \in \mathbb{Z}\}$ and $c^{2\mathbb{Z}+1} = \{c^{2n+1} : n \in \mathbb{Z}\}$ are disjoint. The union $c^{2\mathbb{Z}} \cup c^{2\mathbb{Z}+1}$ coincides with the cyclic subgroup $c^\mathbb{Z}$ of H generated by c. Find a set $S \subset X$ that intersects each coset $c^\mathbb{Z}x$, $x \in X$, at a single point. Since $c^{2\mathbb{Z}}$ has infinite index in X, the set S is infinite. Now observe that for every $E \subset S$ the union

$$T_E = c^{2\mathbb{Z}} \cdot E \cup c^{2\mathbb{Z}+1} \cdot (S \setminus E)$$

is a twin set with $c \in \mathrm{Fix}^-(T_E)$. Consequently, $\mathsf{T} \supset \{T_E : E \subset S\}$ has cardinality

$$|\mathsf{T}| \geq |\{T_E : E \subset S\}| \geq |2^S| \geq \mathfrak{c} > \aleph_0.$$

$\qquad\square$

Now we apply the above results to the minimal upper subfamilies T_K with $K \in \widehat{\mathcal{K}}$.

Theorem 16.6. *For a maximal 2-cogroup K of a group X the following conditions are equivalent:*
   (1) $\mathrm{End}_\lambda(\mathsf{T}_K)$ *is metrizable;*
   (2) $\mathrm{End}_\lambda(\mathsf{T}_K)$ *is a semi-topological semigroup;*
   (3) $\mathrm{End}_\lambda(\mathsf{T}_K)$ *is a finite semigroup;*

(4) $\mathrm{End}_\lambda(\mathsf{T}_K)$ is isomorphic to $C_{2^k} \wr m^m$ or $Q_{2^k} \wr m^m$ for some $1 \le k \le m < \infty$;

(5) K has finite index in X;

(6) the family T_K is finite;

(7) $|\mathsf{T}_K| < \mathfrak{c}$.

Proof. The implications $(1) \Rightarrow (2) \Rightarrow (5)$ follow from Theorems 16.4 and 16.1.

$(5) \Rightarrow (6)$ If the 2-cogroup K has finite index m in X, then the subgroup KK also has finite index in X. Consequently, the family

$$\mathsf{T}_K = \{A \subset X : \mathrm{Fix}^-(A) = K\} = \{A \subset X : AKK = A, \ AK = X \setminus A\}$$

is finite and has cardinality $2^{m/2}$.

The implication $(6) \Rightarrow (7)$ is trivial.

$(7) \Rightarrow (5)$ If K has infinite index in X, then the subgroup $K^\pm = K \cup KK$ has infinite index in X. Take any subset $S \subset X$ meeting each coset $K^\pm x$, $x \in X$, at a single point. Since for every $E \subset S$ the twin set $T_E = KKE \cup K\,(S \setminus E)$ belongs to T_K, we see that $|\mathsf{T}_K| \ge 2^{|S|} \ge 2^{\aleph_0} = \mathfrak{c}$.

$(5) \Rightarrow (4)$ Assume that K has finite index in X. Then the characteristic group $H(K)$ of K is finite and hence is isomorphic to C_{2^k} or Q_{2^k} for some $k \in \mathbb{N}$, see Theorem 9.1(2). Also the set T_K is finite and hence $m = |[\mathsf{T}_K]|$ is finite too. By Theorem 15.1(3), the semigroup $\mathrm{End}_\lambda(\mathsf{T}_K)$ is isomorphic to $H(K) \wr [\mathsf{T}_K]^{[\mathsf{T}_K]}$ and the latter semigroup is isomorphic to $C_{2^k} \wr m^m$ or $Q_{2^k} \wr m^m$ for $m = |[\mathsf{T}_K]|$.

The implications $(4) \Rightarrow (3) \Rightarrow (1)$ are trivial. $\square$

17. The structure of the superextensions of twinic groups

In this section we apply Corollaries 14.6, 14.9, and Theorem 15.1 to describing the structure of minimal left ideals of superextensions of twinic groups and obtain:

Theorem 17.1. *Let X be a twinic group and $\mathcal{K}' \subset \widehat{\mathcal{K}}$ be a subfamily intersecting each orbit $[K] = \{xKx^{-1} : x \in X\}$, $K \in \widehat{\mathcal{K}}$, at a single point. Then*

(1) *each minimal left ideal of $\lambda(X)$ is algebraically isomorphic to the product*

$$\prod_{K \in \mathcal{K}'} H(K) \times [\mathsf{T}_K]$$

where each orbit space $[\mathsf{T}_K]$ is endowed with left zero multiplication;

(2) *each maximal subgroup of the minimal ideal $K(\lambda(X))$ of $\lambda(X)$ is algebraically isomorphic to $\prod_{K \in \mathcal{K}'} H(K)$;*

(3) *for any minimal idempotent $\mathcal{E} \in K(\lambda(X))$ the maximal group $H(\mathcal{E}) = \mathcal{E} \circ \lambda(X) \circ \mathcal{E}$ that contains $\mathcal{E}$ is topologically isomorphic to $\prod_{K \in \mathcal{K}'} H(A_K)$ where $A_K \in \Phi_\mathcal{E}(\mathsf{T}_K)$ for $K \in \mathcal{K}'$;*

(4) *If the family $\widehat{\mathsf{T}}$ is maximally $\mathcal{I}$-independent, then the semigroup $\lambda(X)$ contains a principal left ideal, which is algebraically isomorphic to $\prod_{K \in \mathcal{K}'} H(K) \wr [\mathsf{T}_K]^{[\mathsf{T}_K]}$.*

For twinic groups X whose all maximal 2-cogroups have finite index the algebraic isomorphisms appearing in Theorem 15.1 are topological isomorphisms. Let us observe that for a 2-cogroup $K \subset X$ of finite index in a group X the characteristic group $H(K)$ is finite and so are the families T_K and $[\mathsf{T}_K]$. We endow these finite spaces with the discrete topology.

Theorem 17.2. *Let X be a twinic group such that each maximal 2-cogroup $K \in \widehat{\mathcal{K}}$ has finite index in X. Let $\mathcal{K}' \subset \widehat{\mathcal{K}}$ be a subfamily intersecting each orbit $[K] = \{xKx^{-1} : x \in X\}$, $K \in \widehat{\mathcal{K}}$, at a single point. Then*

(1) *each minimal left ideal of $\lambda(X)$ is topologically isomorphic to the compact topological semigroup $\prod_{K \in \mathcal{K}'} H(K) \times [\mathsf{T}_K]$ where each orbit space $[\mathsf{T}_K]$ is endowed with a left zero multiplication;*

(2) *each maximal subgroup of the minimal ideal $K(\lambda(X))$ of $\lambda(X)$ is topologically isomorphic to the compact topological group $\prod_{K \in \mathcal{K}'} H(K)$;*

(3) *The semigroup $\lambda(X)$ contains a principal left ideal, which is topologically isomorphic to the compact topological semigroup $\prod_{K \in \mathcal{K}'} H(K) \wr [\mathsf{T}_K]^{[\mathsf{T}_K]}$.*

This theorem can be written in a form, more convenient for calculations.

Given a group H let $\eta(X, H)$ denote the number of all orbits $[K] \in [\widetilde{\mathcal{K}}]$ such that for some (equivalently, every) 2-cogroup $K \in [K]$ the characteristic group $H(K)$ is isomorphic to H.

The cardinal number $\eta(X, H)$ can be written as the sum

$$\eta(X, H) = \sum_{1 \leq m \leq |X|} \eta(X, H; m)$$

where $\eta(X, H; m)$ is the number of orbits $[K] \in [\widetilde{\mathcal{K}}]$ such that for every $K \in [K]$ the characteristic group $H(K)$ is isomorphic to H and $|X/K| = m$ where $X/K = \{xK : x \in X\}$.

The following proposition can be easily derived from Theorem 9.1.

Proposition 17.3. *Let X and H be groups. If $\eta(X, H; m) \neq 0$ for some cardinal m, then*

(1) *H is a 2-group with a unique element of order 2;*
(2) *If X is commutative, then H is isomorphic to C_{2^k} for some $1 \leq k \leq \infty$ and $m = |H|$.*
(3) *If m is finite, then the group H is isomorphic to C_{2^k} or Q_{2^k} for some numbers k such that 2^k divides m.*

Theorem 17.2, Proposition 17.3 and Theorem 15.1 imply:

Theorem 17.4. *Let X be a twinic group such that each maximal 2-cogroup $K \in \widehat{\mathcal{K}}$ has finite index in X. Then*

(1) *each minimal left ideal of $\lambda(X)$ is topologically isomorphic to the compact topological semigroup*

$$\prod_{1 \leq k \leq m < \infty} (C_{2^k} \times m_k)^{\eta(X, C_{2^k}; m)} \times (Q_{2^k} \times m_k)^{\eta(X, Q_{2^k}; m)}$$

where each cardinal $m_k = 2^{m/2-k}$ is endowed with left zero multiplication;
(2) *each maximal subgroup of the minimal ideal $K(\lambda(X))$ of $\lambda(X)$ is topologically isomorphic to the compact topological group*

$$\prod_{1 \leq k < \infty} (C_{2^k})^{\eta(X, C_{2^k})} \times (Q_{2^k})^{\eta(X, Q_{2^k})}.$$

(3) *the semigroup $\lambda(X)$ contains a principal left ideal that is topologically isomorphic to the compact topological semigroup*

$$\prod_{1 \leq k \leq m < \infty} (C_{2^k} \wr m_k^{m_k})^{\eta(X, C_{2^k}; m)} \times (Q_{2^k} \wr m_k^{m_k})^{\eta(X, Q_{2^k}; m)}.$$

In light of Theorem 17.4 the following problem seems to be important.

Problem 17.5. *Given a groups X calculate the cardinal numbers $\eta(X, C_{2^k}; m)$ and $\eta(X, Q_{2^k}; m)$ for finite numbers k, m.*

For $m = 2^k$ this problem can be easily answered. For a group H let $\hom(X, H)$ denote the set of all homomorpisms from X into H and $\mathrm{epi}(X, H)$ be its subset consisting of surjective homomorphisms from X onto H.

Proposition 17.6. *For any integer $k \geq 1$ and $n \geq 3$ we get*

(1) *$\eta(X, C_{2^k}; 2^k) = |\mathrm{epi}(X, C_{2^k})|/2^{k-1} = |\hom(X, C_{2^k}) \setminus \hom(X, C_{2^{k-1}})|/2^{k-1}$;*
(2) *$\eta(X, Q_{2^n}; 2^n) = |\mathrm{epi}(X, Q_{2^n})|/|\mathrm{Aut}(Q_{2^n})|$;*
(3) *$|\mathrm{Aut}(Q_8)| = 24$ and $|\mathrm{Aut}(Q_{2^{n+1}})| = 2^{2n-1}$;*
(4) *$|\mathrm{epi}(X, Q_8)| = |\hom(X, Q_8)| - 3|\hom(X, C_4)| + 2|\hom(X, C_2)|$;*
(5) *If $|\hom(X, Q_{2^{n+1}}| < \infty$ then*

$$|\mathrm{epi}(X, Q_{2^{n+1}})| = |\hom(X, Q_{2^{n+1}})| - 2|\hom(X, Q_{2^n})| - |\hom(X, C_{2^n})| + 2|\hom(X, C_{2^{n-1}})|.$$

Proof. 1. By definition, the number $\eta(X, C_{2^k}; 2^k)$ is equal to the number of orbits $[K] \in [\widehat{\mathcal{K}}]$ such that for every $K \in [K]$ the characteristic group $H(K) = \mathrm{Stab}(K)/KK$ is isomorphic to C_{2^k} and $|X/KK| = 2^k$. This implies that $\mathrm{Stab}(K) = X$ and hence the subgroup KK is normal in X. It follows that the group KK coincides with the kernel of some epimorphism $f : X \to C_{2^k}$ while $K = f^{-1}(-1)$ (we recall that $C_{2^k} = \{z \in \mathbb{C} : z^{2^k} = 1\}$). Observe that two epimorphisms $f, g : X \to C_{2^k}$ have the same kernel if and only if $f^{-1}(-1) = g^{-1}(-1)$ if and only if $g = \alpha \circ f$ for some automorphism of the group C_{2^k}. The group C_{2^k} has exactly 2^{k-1} automorphisms determined by the image of the generator $a = e^{i\pi 2^{-k+1}}$ of C_{2^k} in the 2-cogroup $aC_{2^{k-1}}$. Consequently,

$$\eta(X, C^{2^k}; 2^k) = |\mathrm{epi}(X, C_{2^k})|/|\mathrm{Aut}(C_{2^k})| = |\mathrm{epi}(X, C_{2^k})|/2^{k-1}.$$

Observe that a homomorphism $f : X \to C_{2^k}$ is surjective if and only if $f(X) \not\subset C_{2^{k-1}}$. Consequently, $|\mathrm{epi}(X, C_{2^k})| = |\hom(X, C_{2^k}) \setminus \hom(X, C_{2^{k-1}})|$.

2. Repeating the argument of the preceding item, we can show that $\eta(X, Q_{2^n}; 2^n) = |\mathrm{epi}(X, Q_{2^n})|/|\mathrm{Aut}(Q_{2^n})|$.

3. To see that $|\mathrm{Aut}(Q_8)| = 24$ observe that the elements $\mathbf{i}, \mathbf{j}$ generate the quaternion group Q_8. A map $f : \{\mathbf{i}, \mathbf{j}\} \to Q_8 \setminus \{1, -1\}$ uniquely extends to an automorphism of Q_8 if and only if $f(\mathbf{i}) \neq \pm f(\mathbf{j})$. This implies that $|\mathrm{Aut}(Q_8)| = 6 \cdot 4 = 24$.

Next, we show that $|\mathrm{Aut}(Q_{2^{n+1}})| = 2^{2n-1}$. We recall that the group $Q_{2^{n+1}}$ has presentation $\langle x, y \mid x^2 = y^{2^{n-1}}, x^4 = 1, xyx^{-1} = y^{-1}\rangle$. Identify the cyclic subgroup generated by the element y with C_{2^n}. Observe that a map $f : \{x, y\} \to Q_{2^{n+1}}$ extends to an automorphism of $Q_{2^{n+1}}$ if and only if $f(y) \in C_{2^n} \setminus C_{2^{n-1}}$ and $f(x) = Q_{2^{n+1}} \setminus C_{2^n}$. This implies that

$$|\mathrm{Aut}(Q_{2^{n+1}})| = |C_{2^n} \setminus C_{2^{n-1}}| \cdot |Q_{2^{n+1}} \setminus C_{2^n}| = 2^{n-1} \cdot 2^n = 2^{2n-1}.$$

4. The quaternion group Q_8 has 3 maximal proper subgroups $H_1 = \{\pm 1, \pm \mathbf{i}\}$, $H_2 = \{\pm 1, \pm \mathbf{j}\}$, $H_3 = \{\pm 1, \pm \mathbf{k}\}$. These cyclic subgroups intersect by the subgroup $C_2 = \{1, -1\}$. Consequently, $\mathrm{epi}(X, Q_8) = \hom(X, Q_8) \setminus \bigcup_{i=1}^3 \hom(X, H_i)$. Since $\hom(X, H_i) \cap \hom(X, H_j) = \hom(X, C_2)$, we get

$$\left| \bigcup_{i=1}^3 \hom(X, H_i) \right| = 3|\hom(X, C_4)| - 2|\hom(X, C_2)|$$

and

$$|\mathrm{epi}(X, Q_8)| = |\hom(X, Q_8)| - 3|\hom(X, C_4)| + 2|\hom(X, C_2)|.$$

5. Observe that the generalized quaternion group $Q_{2^{n+1}}$ has 3 maximal proper subgroups: $H_0 = C_{2^n}$, $H_1 = C_{2^{n-1}} \cup \mathbf{j}C_{2^{n-1}}$, and $H_2 = C_{2^{n-1}} \cup \mathbf{j}aC^{2^{n-1}}$ where $a \in C_{2^n} \setminus C_{2^{n-1}}$. The subgroups H_1 and H_2 are isomorphic to $Q_{2^{n-1}}$ and $H_i \cap H_j = C_{2^{n-1}}$ for any distinct indices i, j. It follows that

$$|\mathrm{epi}(X, Q_{2^{n+1}})| =$$

$$|\hom(X, Q_{2^{n+1}})| - \sum_{i=0}^2 |\hom(X, H_i) \setminus \hom(X, C_{2^{n-1}})| + |\hom(X, C_{2^{n-1}})| =$$

$$|\hom(X, Q_{2^{n+1}})| - 2|\hom(X, Q_{2^n})| - |\hom(X, C_{2^n})| + 2|\hom(X, C_{2^{n-1}})|.$$

$\square$

The numbers $\eta(X, C_{2^k})$ can be effectively calculated for any finitely generated group X. In this case the group X can be written as the direct sum $\oplus_{\alpha \in A} G_\alpha$ of cyclic groups such that each cyclic group G_α is either infinite or a p-group for some prime number p. For $n \in \mathbb{N} \cup \{\infty\}$ let $r_n(X) = |\{\alpha \in A : |G_\alpha| = n\}|$. It is clear that $r_n(X) = 0$ for all sufficiently large numbers $n \in \mathbb{N}$.

Proposition 17.7. *If X is a finitely generated abelian group, then for every $k \in \mathbb{N}$*

(1) $|\hom(X, C_{2^k})| = 2^{k \cdot r_\infty(X)} \cdot \prod_{n=1}^\infty 2^{\min\{n,k\} \cdot r_{2^n}(X)}$;

(2) $\eta(X, C_{2^k}) = (|\hom(X, C_{2^k})| - |\hom(X, C_{2^{k-1}})|)/2^{k-1}$.

Proof. 1. Let $X = \oplus_{\alpha \in A} G_\alpha$ where each subgroup G_α of X is either infinite cyclic or a cyclic p-group for some prime number p. It follows that the group $\hom(X, C_{2^k})$ of homomorphisms from X to C_{2^k} can be identified with the product $\prod_{\alpha \in A} \hom(G_\alpha, C_{2^k})$. It is clear that $\hom(\mathbb{Z}, C_{2^k}) \cong C_{2^k}$ while $\hom(C_{2^n}, C_{2^k}) \cong C_{2^{\min\{n,k\}}}$ where $\cong$ means "isomorphic to". Now we see that

$$\hom(X, C_{2^k}) \cong \prod_{\alpha \in A} \hom(G_\alpha, C_{2^k}) \cong$$

$$\hom(\mathbb{Z}, C_{2^k})^{r_X(\infty)} \times \prod_{n=1}^{\infty} \hom(C_{2^n}, C_{2^k})^{r_X(2^n)} \cong$$

$$C_{2^k}^{r_X(\infty)} \times \prod_{n=1}^{\infty} C_{2^{\min\{n,k\}}}^{r_X(2^n)}$$

which implies the desired equality from the item (1).

2. The second item follows from Proposition 17.7(1). $\qquad\square$

18. Compact reflexions of groups

Till this moment our strategy in describing the minimal left ideals of the semigroups $\lambda(X)$ was finding a relatively small subfamily $\mathsf{F} \subset \mathsf{P}(X)$ such that the function representation $\Phi_\mathsf{F} : \lambda(X) \to \mathrm{End}_\lambda(\mathsf{F})$ is injective on all minimal left ideals of $\lambda(X)$. Now we shall simplify the group X preserving the minimal left ideals of $\lambda(X)$ unchanged.

We shall describe three such simplifying procedures. One of them is the factorization of X by the subgroup

$$\mathrm{Odd} = \bigcap_{K \in \widehat{\mathcal{K}}} KK.$$

Here we assume that $\mathrm{Odd} = X$ if the set $\widehat{\mathcal{K}}$ is empty.

The following proposition explains the choice of the notation for the subgroup Odd. We recall that a group G is called *odd* if each element of G has odd order.

Proposition 18.1. Odd *is the largest normal odd subgroup of* X. *If* X *is Abelian, then* Odd *coincides with the set of all elements having odd order in* X.

Proof. The normality of the subgroup $\mathrm{Odd} = \bigcap_{K \in \widehat{\mathcal{K}}} KK$ follows from the fact that $xKx^{-1} \in \widehat{\mathcal{K}}$ for every $K \in \widehat{\mathcal{K}}$ and $x \in X$. Next, we show that the group Odd is odd. Assuming the converse, we could find an element $a \in \mathrm{Odd}$ such that the sets $a^{2\mathbb{Z}} = \{a^{2n} : n \in \mathbb{Z}\}$ and $a^{2\mathbb{Z}+1} = \{a^{2n+1} : n \in \mathbb{Z}\}$ are disjoint. Then the 2-cogroup $a^{2\mathbb{Z}+1}$ of X can be enlarged to a maximal 2-cogroup $K \in \widehat{\mathcal{K}}$. Then $a \in K \subset X \setminus KK$ and thus $a \notin \mathrm{Odd}$, which is a contradiction.

It remains to prove that Odd contains any normal odd subgroup $H \subset X$. It suffices to check that for every maximal 2-cogroup $K \in \widehat{\mathcal{K}}$ the subgroup $H \subset X$ lies the group KK. Let $K^{\pm} = K \cup KK$. Since the subgroup H is normal in X, the sets $KKH = HKK$ and $K^{\pm}H = HK^{\pm}$ are subgroups. We claim that the sets $KH = HK$ and $KKH = HKK$ are disjoint. In the opposite case H would intersect the set K. Take any point $x \in H \cap K$ and consider the cyclic subgroup $x^{2\mathbb{Z}} = \{x^{2n} : n \in \mathbb{Z}\}$. Since $x \in K$, the subgroup $x^{2\mathbb{Z}}$ does not intersect the set $x^{2\mathbb{Z}+1} = \{x^{2n+1} : n \in \mathbb{Z}\}$. On the other hand, since H is odd, there is an integer number $n \in \mathbb{Z}$ with $x^{2n+1} = x^0 \in x^{2\mathbb{Z}+1} \cap x^{2\mathbb{Z}}$. This contradiction shows that KH and KKH are disjoint. Consequently, the subgroup KKH has index 2 in the group $K^{\pm}H$ and hence $KH = K^{\pm}H \setminus KKH$ is a 2-cogroup in X containing H. The maximality of K in $\mathcal{K}$ guarantees that $K = KH$ and hence $H \subset KK$. $\quad\square$

The quotient homomorphism $q_{\mathrm{odd}} : X \to X/\mathrm{Odd}$ generates a continuous semigroup homomorphism $\lambda(q_{\mathrm{odd}}) : \lambda(X) \to \lambda(X/\mathrm{Odd})$.

The following theorem was proved in [4, 3.3].

Theorem 18.2. *The homomorphism* $\lambda(q_{\mathrm{odd}}) : \lambda(X) \to \lambda(X/\mathrm{Odd})$ *is injective on each minimal left ideal of* $\lambda(X)$.

Next, we define two compact topological groups called the first and second profinite reflexions of the group X. To define the first profinite reflexion, consider the family $\mathcal{N}$ of all normal subgroups of X with finite index in X. For each subgroup $H \in \mathcal{N}$ consider the quotient homomorphism $q_H : X \to X/H$. The diagonal product of those homomorphisms determines the homomorphism $q : X \to \prod_{H \in \mathcal{N}} X/H$ of X into the compact topological group $\prod_{H \in \mathcal{N}} X/H$. The closure of the image $q(X)$ in $\prod_{H \in \mathcal{N}} X/H$ is denoted by $\bar{X}$ and is called the *profinite reflexion* of X.

The second profinite reflexion $\bar{X}_2$ is defined in a similar way with help of the subfamily $\mathcal{N}_2 = \{\bigcap_{x \in X} xKKx^{-1} : K \in \max \mathcal{K}, \ |X/K| < \aleph_0\}$ of $\mathcal{F}$. The quotient homomorphisms $q_H : X \to X/H$, $H \in \mathcal{N}_2$, compose a homomorphism $q_2 : X \to \prod_{H \in \mathcal{N}_2} X/H$. The closure of the image $q_2(X)$ in $\prod_{H \in \mathcal{N}_2} X/H$ is denoted by $\bar{X}_2$ and is called the *second profinite reflexion* of X. Since $\mathrm{Ker}(q_2) = \bigcap_{H \in \mathcal{N}_2} \bigcap_{x \in X} xKKx^{-1} \supset \mathrm{Odd}$, the homomorphism $q_2 : X \to \bar{X}_2$ factorizes through the group X/Odd in the sense that there is a unique homomorphism $q_{\mathrm{even}} : X/\mathrm{Odd} \to \bar{X}_2$ such that $q_2 = q_{\mathrm{even}} \circ q_{\mathrm{odd}}$.

Thus we get the following commutative diagram:

$$
\begin{array}{ccc}
X & \xrightarrow{\ q_{\mathrm{odd}}\ } & X/\mathrm{Odd} \\
{\scriptstyle q}\downarrow & {\scriptstyle q_2}\searrow & \downarrow{\scriptstyle q_{\mathrm{even}}} \\
\bar{X} & \xrightarrow[\ \mathrm{pr}\ ]{} & \bar{X}_2
\end{array}
$$

Applying to this diagram the functor λ of superextension we get the diagram

$$
\begin{array}{ccc}
\lambda(X) & \xrightarrow{\ \lambda(q_{\mathrm{odd}})\ } & \lambda(X/\mathrm{Odd}) \\
{\scriptstyle \lambda(q)}\downarrow & {\scriptstyle \lambda(q_2)}\searrow & \downarrow{\scriptstyle \lambda(q_{\mathrm{even}})} \\
\lambda(\bar{X}) & \xrightarrow[\ \lambda(\mathrm{pr})\ ]{} & \lambda(\bar{X}_2)
\end{array}
$$

In this diagram $\lambda(\bar{X})$ and $\lambda(\bar{X}_2)$ are the superextensions of the compact topological groups $\bar{X}$ and $\bar{X}_2$. We recall that the superextension $\lambda(K)$ of a compact Hausdorff space K is the closed subspace of the second exponent $\exp(\exp(K))$ that consists of the maximal linked systems of closed subsets of K, see [18, §2.1.3].

Theorem 18.3. *If each maximal 2-cogroup K of a twinic group X has finite index in X, then the homomorphism $\lambda(q_2) : \lambda(X) \to \lambda(\bar{X}_2)$ is injective on each minimal left ideal of $\lambda(X)$.*

Proof. The injectivity of the homomorphism $\lambda(q_2)$ on a minimal left ideal L of $\lambda(X)$ will follow as soon as for any distinct maximal linked systems $\mathcal{A}, \mathcal{B} \in \mathsf{L}$ we find a subgroup $H \in \mathcal{N}_2$ such that $\lambda q_H(\mathcal{A}) \neq \lambda q_H(\mathcal{B})$.

By Corollary 14.6, the function representation $\Phi_{\widetilde{\mathsf{T}}} : \lambda(X) \to \mathrm{End}_\lambda(\widehat{\mathsf{T}})$ is injective on the minimal left ideal L. Let $\mathcal{K}' \subset \widehat{\mathcal{K}}$ be a subfamily meeting each orbit $[K]$, $K \in \widehat{\mathcal{K}}$, at a single point. By Corollary 14.9, the semigroup $\mathrm{End}_\lambda(\widehat{\mathsf{T}})$ is topologically isomorphic to the product $\prod_{K \in \mathcal{K}'} \mathrm{End}_\lambda(\mathsf{T}_K)$. Consequently, for some maximal 2-cogroup $K \in \mathcal{K}'$ the images of the maximal linked systems $\mathcal{A}, \mathcal{B}$ under the function representation $\Phi_{\mathsf{T}_K} : \lambda(X) \to \mathrm{End}_\lambda(\mathsf{T}_K)$ are distinct. This means that for some set $T \in \mathsf{T}_K$ there is $x \in X$ such that $x^{-1}T \in \mathcal{A} \notin \mathcal{B}$.

Since the 2-cogroup K has finite index in X, the normal subgroup $H = \bigcap_{x \in X} xKKx^{-1}$ has finite index in X and belongs to the family $\mathcal{N}_2$. Consider the finite quotient group X/H and let $q_H : X \to X/H$ be the quotient homomorphism. Since $H \subset KK$, the set $T = KKT$ coincides with the preimage $q_H^{-1}(T')$ of some twin set $T' \in X/H$. Then the images $\mathcal{A}' = \lambda q_H(\mathcal{A})$ and $\mathcal{B}' = \lambda q_H(\mathcal{B})$ are distinct because for the point $y = q_H(x)$ we get $y^{-1}T' \in \mathcal{A}' \setminus \mathcal{B}'$. $\qquad\square$

Remark 18.4. For each finite abelian group X the group X/Odd is a 2-group. For non-commutative groups it is not true anymore: for the group $X = A_4$ of even permutations of the set $4 = \{0, 1, 2, 3\}$ the group X/Odd coincides with X. Also X/Odd coincides with X for any simple group.

19. Some examples

Now we consider the superextensions of some concrete groups.

19.1. **The superextension of the cyclic group** $\mathbb{Z}$. In order to compare the algebraic properties of the semigroups $\lambda(\mathbb{Z})$ and $\beta(\mathbb{Z})$ let us recall a deep result of E.Zelenyuk [19] (see also [11, §7.1]) who proved that the subsemigroup $\beta(\mathbb{Z}) \subset \lambda(\mathbb{Z})$ of ultrafilters contains no finite subgroup. It turns out that the semigroup $\lambda(\mathbb{Z})$ has totally different property.

Theorem 19.1. (1) *The semigroup $\lambda(\mathbb{Z})$ contains a principal left ideal topologically isomorphic to* $\prod_{k=1}^{\infty} C_{2^k} \wr m_k^{m_k}$ *where $m_k = 2^{2^{k-1}-k}$.*

(2) *Each minimal left ideal of $\lambda(\mathbb{Z})$ is topologically isomorphic to $2^{\omega} \times \prod_{k=1}^{\infty} C_{2^k}$ where the Cantor cube 2^{ω} is endowed with the left-zero multiplication.*

(3) *each maximal group of the minimal ideal $K(\lambda(\mathbb{Z}))$ is topologically isomorphic to $\prod_{k=1}^{\infty} C_{2^k}$.*

(4) *The semigroup $\lambda(\mathbb{Z})$ contains a topologically isomorphic copy of each second countable profinite topological semigroup.*

Proof. The group is abelian and hence has trivial twinic ideal according to Theorem 6.2. It is easy to see that $\eta(\mathbb{Z}, C_{2^k}) = 1$ for all $k \in \mathbb{N}$, while $\eta(\mathbb{Z}, C_{2^{\infty}}) = 0$.

1. By Theorem 17.4 and Proposition 17.3, the semigroup $\lambda(\mathbb{Z})$ contains a principal left ideal that is topologically isomorphic to $\prod_{k=1}^{\infty} C_{2^k} \wr m_k^{m_k}$ where $m_k = 2^{2^{k-1}-k}$.

2. By Theorem 17.4(3), each minimal left ideal $\mathcal{I}$ of $\lambda(\mathbb{Z})$ is topologically isomorphic to $\prod_{k=1}^{\infty} C_{2^k} \times m_k$ where each cardinal $m_k = 2^{2^{k-1}-k}$ is endowed with the left zero multiplication. It is easy to see that the left zero semigroup $\prod_{k=1}^{\infty} m_k$ is topologically isomorphic to the Cantor cube 2^{ω} endowed with the left zero multiplication. Consequently, $\mathcal{I}$ is topologically isomorphic to $2^{\omega} \times \prod_{k=1}^{\infty} C_{2^k}$.

3. The preceding item implies that each maximal group of the minimal ideal $K(\lambda(\mathbb{Z}))$ is topologically isomorphic to $\prod_{k=1}^{\infty} C_{2^k}$.

4. The fourth item follows from the first item and the following well-known fact, see [5, I.1.3]. $\square$

Lemma 19.2. *Each semigroup S is algebraically isomorphic to a subsemigroup of the semigroup A^A of all self-maps of a set A of cardinality $|A| \geq S^1$ where S^1 is S with attached unit.*

19.2. **The superextension of cyclic 2-groups** C_{2^n}. For a cyclic 2-group $X = C_{2^n}$ the number

$$\eta(X, C_{2^k}) = \begin{cases} 1 & \text{if } k \leq n \\ 0 & \text{otherwise.} \end{cases}$$

Applying Theorem 17.4, Proposition 17.3, and Lemma 19.2, we get:

Corollary 19.3. *For every $n \in \mathbb{N}$*

(1) *The semigroup $\lambda(C_{2^n})$ contains a principal left ideal isomorphic to $\prod_{k=1}^{n} C_{2^k} \wr m_k^{m_k}$ where $m_k = 2^{2^{k-1}-k}$.*

(2) *Each minimal left ideal of $\lambda(C_{2^n})$ is isomorphic to $\prod_{k=1}^{n} C_{2^k} \times m_k$ where cardinal $m_k = 2^{2^{k-1}-k}$ is endowed with left-zero multiplication.*

(3) *each maximal group of the minimal ideal $K(\lambda(C_{2^n}))$ is isomorphic to $\prod_{k=1}^{n} C_{2^k}$.*

(4) *The semigroup $\lambda(C_{2^n})$ contains an isomorphic copy of each semigroup S of cardinality $|S| < 2^{2^{n-1}-n}$.*

19.3. **The superextension of the quasi-cyclic 2-group** $C_{2^{\infty}}$. The superextension $\lambda(C_{2^{\infty}})$ has even more interesting properties.

Theorem 19.4. (1) *Minimal left ideal of the semigroup $\lambda(C_{2^{\infty}})$ are not topological semigroups.*

(2) *each minimal left ideal of $\lambda(X)$ is algebraically isomorphic to $\mathfrak{c} \times (C_{2^{\omega}})^{\omega}$ where the cardinal $\mathfrak{c} = 2^{\aleph_0}$ is endowed with left zero multiplication;*

(3) *the semigroup $\lambda(X)$ contains a principal left ideal, which is algebraically isomorphic to $(C_{2^{\omega}} \wr \mathfrak{c}^{\mathfrak{c}})^{\omega}$;*

(4) *$\lambda(X)$ contains an isomorphic copy of each semigroup of cardinality $\leq \mathfrak{c}$;*

(5) *each maximal subgroup of the minimal ideal $K(\lambda(C_{2^\infty}))$ of $\lambda(X)$ is algebraically isomorphic to $(C_{2^\omega})^\omega$;*

(6) *each maximal subgroup of the minimal ideal $K(\lambda(C_{2^\infty}))$ is topologically isomorphic to the countable product $\prod_{n=1}^\infty (C_{2^\infty}, \tau_n)$ of quasi-cyclic 2-groups endowed with twin-generated topologies;*

(7) *for any twin-generated topologies τ_n, $n \in \mathbb{N}$, on C_{2^∞} the right-topological group $\prod_{n=1}^\infty (C_{2^\infty}, \tau_n)$ is topologically isomorphic to a maximal subgroup of $K(\lambda(C_{2^\infty}))$.*

Proof. Since each proper subgroup of C_{2^∞} is finite, the family $\widehat{\mathcal{K}}$ of maximal 2-cogroups is countable and hence can be enumerated as $\widehat{\mathcal{K}} = \{K_n : n \in \omega\}$. For every $n \in \omega$ the maximal 2-cogroup $K \in \widehat{\mathcal{K}}$ has infinite index and its characteristic group $H(K)$ is isomorphic to C_{2^∞}.

1. By Corollaries 14.6 and 14.9, each minimal left ideal of the superextension $\lambda(C_{2^\infty})$ is topologically isomorphic to a minimal left ideal of the product $\prod_{K \in \widehat{\mathcal{K}}} \mathrm{End}_\lambda(\mathsf{T}_K)$. For each maximal 2-cogroup $K \in \widehat{\mathcal{K}}$ each minimal left ideal L of the semigroup $\mathrm{End}_\lambda(\mathsf{T}_K)$ is compact but any maximal subgroup of L, being algebraically isomorphic to C_{2^∞}, is not compact. This implies that L is not a topological semigroup. Then minimal left ideals of the semigroup $\lambda(C_{2^\infty})$ are not topological semigroups neither.

2,3. The statements (2) and (3) follow from Theorem 17.1.

4. The forth item follows from the third one because each semigroup S of cardinality $|S| \leq \mathfrak{c}$ embeds into the semigroup $\mathfrak{c}^{\mathfrak{c}}$ according to Lemma 19.2.

5. By Theorem 17.1(2) each maximal subgroup of the minimal ideal $K(\lambda(C_{2^\infty}))$ is algebraically isomorphic to $(C_{2^\infty})^\omega$.

6. By Theorem 17.1, each maximal subgroup G in the minimal ideal $K(\lambda(C_{2^\infty}))$ is topologically isomorphic to the product $\prod_{K \in \widehat{\mathcal{K}}} H(A_K)$ of the structure groups of suitable twin subsets $A_K \in \mathsf{T}_K = \mathsf{T}_{[K]}$, $K \in \widehat{\mathcal{K}}$. For each maximal 2-cogroup $K \in \widehat{\mathcal{K}}$ the structure group $H(A_K)$ is just C_{2^∞} endowed with a twin-generated topology.

7. Now assume conversely that τ_n, $n \in \mathbb{N}$, are twin generated topologies on the quasi-cyclic group C_{2^∞}. For every $n \in \mathbb{N}$ find a twin subset $A_n \in \mathsf{T}_{K_n}$ whose structure group $H(A_n)$ is topologically isomorphic to (C_{2^∞}, τ_n). By Theorem 17.1(3) and Theorem 15.1(8), the product $\prod_{n=1}^\infty H(A_n)$ is topologically isomorphic to some maximal subgroup of $K(\lambda(C_{2^\infty}))$. $\square$

Remark 19.5. Theorem 17.1(3) and Proposition 6.8 implies that among maximal subgroup of the minimal ideal of $\lambda(C_{2^\infty})$ there are:

- Raikov complete topological groups;
- incomplete totally bounded topological groups;
- paratopological groups, which are not topological groups;
- semi-topological groups, which are not paratopological groups.

20. Superextensions of finite groups

Theorem 17.4 and Proposition 17.7 give us an algorithmic way of calculating the minimal left ideals of the superextensions of finitely-generated abelian groups. For non-abelian groups the situation is a bit more complicated. In this section we shall describe the minimal left ideals of finite groups X of order $|X| \leq 15$.

In fact, Theorem 18.2 helps to reduce the problem to studying superextensions of groups X/Odd. The group X/Odd is trivial if the order of X is odd. So, it suffices to check non-abelian groups of even order. If X is a 2-group, then the subgroup Odd of X is trivial and hence $X/\mathrm{Odd} = X$. Also the subgroup Odd is trivial for simple groups.

The next table describes the structure of minimal left ideals of the superextensions of groups $X = X/\mathrm{Odd}$ of order $|X| \leq 15$. In this table $\mathcal{E}$ stands for a minimal idempotent of $\lambda(X)$, which generates the principal left ideal $\lambda(X) \circ \mathcal{E}$ and lies in the maximal subgroup $H(\mathcal{E}) = \mathcal{E} \circ \lambda(X) \circ \mathcal{E}$. Below the cardinals 2^n are considered as semigroups of left zeros.

X	$|E(\lambda(X) \circ \mathcal{E})|$	$\mathcal{E} \circ \lambda(X) \circ \mathcal{E}$	$\lambda(X) \circ \mathcal{E}$
C_2	1	C_2	C_2
C_4	1	$C_2 \times C_4$	$C_2 \times C_4$
C_2^2	1	C_2^3	C_2^3
C_2^3	1	C_2^7	C_2^7
$C_2 \oplus C_4$	1	$C_2^2 \times C_4^2$	$C_2^3 \times C_4^2$
C_8	2	$C_2 \times C_4 \times C_8$	$2 \times C_2 \times C_4 \times C_8$
D_8	2	C_2^5	$2 \times C_2^5$
Q_8	2	$C_2^3 \times Q_8$	$2 \times C_2^3 \times Q_8$
A_4	2^6	C_2^3	$2^6 \times C_2^3$

For abelian groups the entries of this table are calculated with help of Theorem 17.4 and Proposition 17.7. Let us illustrate this on the following example.

20.1. **The group $C_2 \oplus C_4$.** By Proposition 17.7 for the group $X = C_2 \oplus C_4$ we get

- $\eta(X, C_2) = |\hom(X, C_2)| - |\hom(X, C_1)| = 2 \cdot 2 - 1 = 3$;
- $\eta(X, C_4) = \frac{1}{2}(|\hom(X, C_4)| - |\hom(X, C_2)|) = \frac{1}{2}(2 \cdot 4 - 2 \cdot 2) = 2$;
- $\eta(X, C_{2^k}) = 0$ for $k > 2$.

Then each minimal left ideal of $\lambda(C_2 \oplus C_4)$ is isomorphic to

$$(C_2 \times m_1)^{\eta(X,C_2)} \times (C_4 \times m_2)^{\eta(X,C_4)} = (C_2 \times 2^{2^{1-1}-1})^3 \times (C_4 \times 2^{2^{2-1}-2})^2 = C_2^3 \times C_4^2.$$

Next, we consider the non-abelian groups.

20.2. **The dihedral group D_8.** This group has a presentation

$$\langle a, b \mid b^4 = a^2 = 1, \ aba^{-1} = b^{-1} \rangle.$$

It contains 3 subgroups of order 4:

$$C_4 = \{1, b, b^2, b^3\}, \quad H_1 = \{1, b^2, a, ab^2\}, \quad H_2 = \{1, b^2, ab, ab^3\},$$

and 5 subgroups of order 2: $C_2 = \{1, b^2\}$, $\{1, a\}$, $\{1, ab\}$, $\{1, ab^2\}$, and $\{1, ab^3\}$. Some of those subgroups correspond to maximal 2-cogroups:

$$K_0 = D_8 \setminus C_4, \ K_1 = D_8 \setminus H_1, \ K_2 = D_8 \setminus H_2, \ K_3 = \{b^2, a\}, \ K_4 = \{b^2, ab\}.$$

Observe that the characteristic groups $H(K_i)$ of those maximal 2-cogroups all are isomorphic to C_2.

On the other hand, for $i \in \{0, 1, 2\}$ the index $|X/K_i| = 2$ and hence $|[\mathsf{T}_{K_i}]| = 2^{2/2-1} = 1$, while for $i \in \{3, 4\}$ we get $|X/K_i| = 4$ and $|[\mathsf{T}_{K_i}]| = 2^{4/2-1} = 2$ by Proposition 15.2(3). Applying Theorem 17.4, we conclude that each minimal left ideal of the superextension $\lambda(D_8)$ is isomorphic to $(C_2 \times 1)^3 \times (C_2 \times 2)^2 = 2 \times C_2^5$.

20.3. **The quaternion group Q_8.** The group $Q_8 = \{\pm 1, \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}\}$ contains 3 cyclic subgroups of order 4 corresponding to 4-element maximal 2-cogroups: $K_1 = Q_8 \setminus \langle \mathbf{i} \rangle$, $K_2 = Q_8 \setminus \langle \mathbf{j} \rangle$, $K_3 = Q_8 \setminus \langle \mathbf{k} \rangle$. The characteristic groups of those 2-cogroups are isomorphic to C_2. The trivial subgroup of Q_8 corresponds to the maximal 2-cogroup $K_0 = \{-1\}$ whose characteristic group coincides with Q_8. By Proposition 15.2(4), for $i \in \{1, 2, 3\}$, we get $|[\mathsf{T}_{K_i}]| = 1$ while $|[\mathsf{T}_{K_0}]| = 2^{8/2-3} = 2$. By Theorem 17.4, each minimal left ideal of the semigroup $\lambda(Q_8)$ is isomorphic to

$$(C_2 \times 1)^3 \times (Q_8 \times 2) = 2 \times C_2^3 \times Q_8.$$

20.4. **The alternating group A_4.** The group A_4 has order 12, contains a normal subgroup isomorphic to $C_2 \times C_2$ and contains no subgroup of order 6. This implies that all 2-cogroups of A_4 lie in $C_2 \times C_2$ and consequently, A_4 contains 3 maximal 2-cogroups. Each maximal 2-cogroup $K \subset A_4$ contains two elements and has characteristic group $H(K)$ isomorphic to C_2. Since $|X/K| = 6$, Proposition 15.2(3) guarantees that $|[\mathsf{T}_K]| = 2^{6/2-1} = 2^2$. Applying Theorem 17.4, we see that each minimal left ideal of the semigroup $\lambda(A_4)$ is isomorphic to $(C_2 \times 2^2)^3 = 2^6 \times C_2^3$.

21. Open Problems

Problem 21.1. Describe the structure of (minimal left ideals) of superextensions of the quaternion groups Q_{2^k} for $3 \leq k \leq \infty$.

Problem 21.2. Describe the structure of (minimal left ideals) of superextensions of the finite groups of order 16.

Problem 21.3. What can be said about the structure of the superextension $\lambda(F_2)$ of the free group F_2.

Problem 21.4. Investigate the permanence properties of the class of twinic groups. Is this class closed under taking subgroups and products?

In light of Theorem 9.1 the following problem is quite interesting.

Problem 21.5. Is each infinite 2-group with a unique element of order 2 isomorphic to C_{2^∞} or Q_{2^∞}?

References

[1] S.I. Adian, *Random walks on free periodic groups*, Math. USSR Izvestiya, **21** (1983), 425–434.

[2] T. Banakh, V. Gavrylkiv, O. Nykyforchyn, *Algebra in superextensions of groups, I: zeros and commutativity* // Algebra Discrete Math. no.3 (2008), 1-29.

[3] T. Banakh, V. Gavrylkiv, *Algebra in superextension of groups, II: cancelativity and centers* // Algebra Discrete Math. no.4 (2008) 1-14.

[4] T. Banakh, V. Gavrylkiv, *Algebra in superextension of groups, III: minimal left ideals* // Mat. Stud. 31:2 (2009) 142-148.

[5] A.H. Clifford, G.B. Preston, The algebraic theory of semigroups. Vol. I., Mathematical Surveys. **7**. AMS, Providence, RI, 1961.

[6] G.S. Deryabina, P.A. Kozhevnikov, *The derived subgroup of a group with commutators of bounded order can be non-periodic*, Communications in Algebra, **27**:9 (1999), 4525–4530.

[7] V.Gavrylkiv, *The spaces of inclusion hyperspaces over noncompact spaces* // Matem. Studii. **28:1** (2007), 92–110.

[8] V. Gavrylkiv, *Right-topological semigroup operations on inclusion hyperspaces* // Matem. Studii **29**:1 (2008), 18–34.

[9] N. Hindman, *Finite sums from sequences within cells of partition of* $\mathbb{N}$ // J. Combin. Theory Ser. A **17** (1974), 1–11.

[10] N. Hindman, *Ultrafilters and combinatorial number theory* // Lecture Notes in Math. **751** (1979), 49–184.

[11] N. Hindman, D. Strauss, Algebra in the Stone-Čech compactification, de Gruyter, Berlin, New York, 1998.

[12] M. Kilp, U. Knauer, A. Mikhalev, Monoids, acts and categories. With applications to wreath products and graphs, de Gruyter Expositions in Mathematics, 29. Walter de Gruyter & Co., Berlin, 2000.

[13] J. van Mill, Supercompactness and Wallman spaces, Math Centre Tracts. **85**. Amsterdam: Math. Centrum., 1977.

[14] D. Osin, *Uniform non-amenability of free Burnside groups*, Arch. Math. **88**:5 (2007), 403–412.

[15] A. Paterson, Amenability, MSM, **29**, AMS, Providence, RI, 1988.

[16] I. Protasov, Combinatorics of Numbers, VNTL, Lviv, 1997.

[17] D. Robinson, A course in the theory of groups, Springer, 1996.

[18] A. Teleiko, M. Zarichnyi, Categorical Topology of Compact Hausdofff Spaces, VNTL, Lviv, 1999.

[19] E.G. Zelenyuk, *Finite groups in* $\beta\mathbb{N}$ *are trivial*, Dopov. Nats. Akad. Nauk Ukr. Mat. 1997, no. 7, 12–14 (in Russian).

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